The role of the Weyl function in Krein's theory of semibounded extensions of non-negative operators.

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Theorem 1. (The first representation theorem)

Let $\mathfrak{t}$ be a closed densely defined lower semibounded quadratic form in $\mathfrak{H}$ $(\mathfrak{t} \geqslant \gamma)$. Then there exists an operator $T = T^* \geqslant \gamma$, such that:

(*i*) dom $T \subset$ dom $\mathfrak{t}$ and

$$\mathfrak{t}[u, v] = (Tu, v), \ u \in \text{ dom } T, \quad v \in \text{dom}\, \mathfrak{t}; \tag{1}$$

(*ii*) dom T is a core of the form $\mathfrak{t}$;

(*iii*) If $u \in \text{dom}\, \mathfrak{t}$, $w \in \mathfrak{H}$ and

$$\mathfrak{t}[u, v] = (w, v) \tag{2}$$

is valid for all v belonging to the core of the form $\mathfrak{t}$, then $u \in$ dom T and $Tu = w$.

Definition 2

Let A be a non-negative symmetric operator in $\mathfrak{H}$.
Let us equip the domain $\mathbf{dom}\, A$ with the norm

$$\|f\|_A^2 = \|f\|^2 + (Af, f). \tag{3}$$

and denote by $D[A]$ the Hilbert space obtained after completion.
This space is called the energy space of the operator A.
Since the form given by (3) is closable (Firedrichs' lemma), the space $D[A]$ is (continuously) embedded in $\mathfrak{H}$.

Definition 3

A self-adjoint oper-r associated with the closure $\mathfrak{a}$ of the form $\mathfrak{a}'$,

$$\mathfrak{a}'[u] = (Au, u), \quad u \in \operatorname{dom} A, \tag{4}$$

is called the Friedrichs extension of the operator A and is denoted by $\widehat{A}_F$. It follows that the lower bounds of A and $\widehat{A}_F$ coincide. It is known that $\mathbf{dom}\, \mathfrak{a} = \operatorname{dom}(\widehat{A}_F)^{1/2}$

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Theorem 4 (Krein M.G.)

Let A be a closed semibounded symmetric operator in $\mathfrak{H}$. Then:
(*i*) $D[A] = D[\widehat{A}_F]$;
(*ii*) $\operatorname{dom} \widehat{A}_F = \operatorname{dom} A^* \cap D[A]$;
(*iii*) If $\widetilde{A} \in \mathrm{Ext}_A(0,\infty)$ and $\widetilde{\mathfrak{a}}$ is the form, associated with the operator $\widetilde{A}$, then $\mathfrak{a} \subseteq \widetilde{\mathfrak{a}}$ and, in particular, $D[\widehat{A}_F] \subset D[\widetilde{A}]$;
(*iv*) If $\widetilde{A} \in \mathrm{Ext}_A(0,\infty)$ and $\operatorname{dom} \widetilde{A} \subset D[A]$, then $\widetilde{A} = \widehat{A}_F$.

2.2 Comparison of semibounded forms

Definition 5

Let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be closed semibounded forms and let A_1, A_2 be the selfadjoint oper-s associated with $\mathfrak{a}_1$ and $\mathfrak{a}_2$, resp.. Then:
(i) $\mathfrak{a}_1 \geqslant \mathfrak{a}_2$ if

$$\operatorname{dom} \mathfrak{a}_1 \subseteq \operatorname{dom} \mathfrak{a}_2 \quad \text{and} \quad \mathfrak{a}_1[u] \geqslant \mathfrak{a}_2[u], \quad u \in \operatorname{dom} \mathfrak{a}_1;$$

(ii) $A_1 \geq A_2$, if $\mathfrak{a}_1 \geqslant \mathfrak{a}_2$.

Theorem 6 (Krein M.G.)

The set $\mathrm{Ext}_A(0,\infty)$ of all nonnegative self-adjoint ext-s of A, contains the maximal $\widehat{A}_F$ and minimal $\widehat{A}_K$ extensions, i.e.

$$\widehat{A}_K \leq \widetilde{A} \leq \widehat{A}_F, \quad \widetilde{A} \in \mathrm{Ext}_A(0,\infty) \quad \Longleftrightarrow \tag{5}$$

$$(\widehat{A}_F + x)^{-1} \leq (\widetilde{A} + x)^{-1} \leq (\widehat{A}_K + x)^{-1}, \quad x \in (0,\infty),\ \widetilde{A} \in \mathrm{Ext}_A(0,\infty)$$

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$$(\widehat{A}_F+x)^{-1} \leq (\widetilde{A}+x)^{-1} \leq (\widehat{A}_K+x)^{-1}, \quad x \in (0,\infty),\ \widetilde{A} \in \mathrm{Ext}_A(0,\infty)$$

The operator $\widehat{A}_F$ coincides with the Friedrichs extension. The extension $\widehat{A}_K$ is called the Krein extension.

Theorem 7

Let A be a densely defined nonnegative symmetric operator in $\mathfrak{H}$ and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* such that $A_0 (= A^* \upharpoonright \ker \Gamma_0) \geq 0$. Let also $M(\cdot)$ be the corresponding Weyl function. Then $A_0 = \widehat{A}_K$ ($A_0 = \widehat{A}_F$) if and only if

$$\lim_{x \uparrow 0} (M(x)f, f) = +\infty, \quad f \in \mathcal{H} \setminus \{0\}$$

$$(\lim_{x \downarrow -\infty} (M(x)f, f) = -\infty, \quad f \in \mathcal{H} \setminus \{0\}).$$

Theorem 8, [3]

Let A be a densely defined nonnegative symmetric operator in $\mathfrak{H}$ and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* such that $A_0 (= A^* \upharpoonright \ker \Gamma_0) \geq 0$. Let also $M(\cdot)$ be the corresponding Weyl function. Then:

(i) There exist strong resolvent limits

$$M(0) := s{-}R{-}\lim_{x \uparrow 0} M(x), \qquad M(-\infty) := s{-}R{-}\lim_{x \downarrow -\infty} M(x). \quad (6)$$

(ii) $M(0)$ and $M(-\infty)$ are self-adjoint linear relations in $\mathcal{H}$ associated with the semibounded below (above) quadratic forms

$$\mathrm{t}_0[f] = \lim_{x \uparrow 0} (M(x)f, f) \geq \beta ||f||^2, \ \ \mathrm{t}_{-\infty}[f] = \lim_{x \downarrow -\infty} (M(x)f, f) \leq \alpha ||f||^2,$$

$$\mathrm{dom}\,(\mathrm{t}_0) = \left\{ f \in \mathcal{H} : \lim_{x \uparrow 0} |(M(x)f, f)| < \infty \right\} = \mathrm{dom}\,((M(0)_{\mathrm{op}} - \beta)^{1/2}),$$

$$\mathrm{dom}\,(\mathrm{t}_{-\infty}) = \left\{ f \in \mathcal{H} : \lim_{x \downarrow -\infty} |(M(x)f, f)| < \infty \right\} = \mathrm{dom}\,((\alpha - M(-\infty)_{\mathrm{op}})^{1/2}).$$

Theorem 8, [3]

Moreover,

$$\mathrm{dom}\,(A_K) = \{f \in \mathrm{dom}\,(A^*) : \ \{\Gamma_0 f, \Gamma_1 f\} \in M(0)\},$$
$$\mathrm{dom}\,(A_F) = \{f \in \mathrm{dom}\,(A^*) : \ \{\Gamma_0 f, \Gamma_1 f\} \in M(-\infty)\}.$$

(iii) Extensions A_0 and A_K are disjoint (A_0 and A_F are disjoint) if and only if

$$M(0) \in \mathcal{C}(\mathcal{H}) \qquad (M(-\infty) \in \mathcal{C}(\mathcal{H}), \ \text{ respectively}).$$

Moreover, in this case

$$\mathrm{dom}\,(A_K) = \mathrm{dom}\,(A^*) \upharpoonright \ker(\Gamma_1 - M(0)\Gamma_0)$$
$$(\mathrm{dom}\,(A_F) = \mathrm{dom}\,(A^*) \upharpoonright \ker(\Gamma_1 - M(-\infty)\Gamma_0), \ \text{ respectively}).$$

Theorem 9

Let A be a closed densely defined nonnegative symmetric operator in $\mathfrak{H}$, $A \geq m_A I \geq 0$, and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^* such that $A_0 = \widehat{A}_F$. Let also Θ is semibounded self-adjoint linear relations in $\mathcal{H}$ and $a < 0$. Then:
(i) $A_\Theta \geq a, \iff \Theta - M(a) \geq 0$;
(ii) $A_\Theta \geq 0 \iff \mathfrak{t}_\Theta - \mathfrak{t}_{M(0)} \geq 0$. In particular, $\operatorname{dom} \mathfrak{t}_\Theta \subset \operatorname{dom} \mathfrak{t}_{M(0)}$.
(iii) The total multiplicities of the negative spectra of A_Θ and $\Theta - M(0)$ coincide:

$$\kappa_-(A_\Theta) := \dim \operatorname{ran}(E_{A_\Theta}(\mathbb{R}_-)) = \kappa_-(\mathfrak{t}_\Theta - \mathfrak{t}_{M(0)}). \tag{7}$$

(iv) If $m_A > 0$, then A_Θ is positive definite if and only if Θ is;
(v)For any $p \in (0, \infty]$, the following equivalence holds:

$$E_{\Theta - M(a)}(\mathbb{R}_-)(\Theta - M(a)) \in \mathfrak{S}_p \iff E_{A_\Theta}(-\infty, a)A_\Theta \in \mathfrak{S}_p.$$

For $a = 0$ the equivalence is replaced by the implication $\Longrightarrow$.

Theorem 9

(vi) For every $\gamma \in (0,\infty)$ the following equivalence holds

$$\lambda_j(A_\Theta) = j^{-\gamma}(a + o(1)) \iff \lambda_j(\Theta(0)) = j^{-\gamma}(b + o(1))$$

as $j \to \infty$. Moreover, either $ab \neq 0$ or $a = b = 0$.

Here $\mathfrak{t}_\Theta$ and $\mathfrak{t}_{M(0)}$ denote the closed quadratic forms associated with the relations Θ and $M(0)$, respectively, in accordance with the first representation theorem.

Corollary 10

Assume the conditions of Theorem 9. If $\mathrm{dom}\,\Theta \subset \mathrm{dom}\,(M(0))$, in particular, if $M(0) \in \mathcal{B}(\mathcal{H})$, then the equivalence holds

$$A_\Theta \geq 0 \iff \Theta - M(0) \geq 0.$$

Matrix Sturm-Liouville operator

Let $A := A_{\min}$ — be the minimal operator, generated in $L^2(\mathbb{R}_+, \mathbb{C}^m)$ by the differential expression $\mathcal{A} := -\frac{d^2}{dx^2} + Q(x), \quad Q = Q^* \in L^2(\mathbb{R}_+, \mathbb{C}^{m\times m})$. Then $A_{\max} = A^*$. Moreover,

$$\operatorname{dom}(A_{\min}) = H_0^2(\mathbb{R}_+, \mathbb{C}^m), \quad \operatorname{dom}(A_{\max}) = H^2(\mathbb{R}_+, \mathbb{C}^m). \quad (8)$$

The closure $\mathfrak{t}_{A_B}$ of the quadratic form $\mathfrak{t}'_{A_B}$ is given by

$$\mathfrak{t}_{A_B}[f] = \int_0^\infty \left(|f'(x)| + Q(x)|f(x)|\right)^2 dx + B|f(0)|^2, \qquad \operatorname{dom}\mathfrak{t}_A = H_0^1(\mathbb{R}_+, \mathbb{C}^m), \quad (9)$$

where $\operatorname{dom} A_B = \{f \in \operatorname{dom} A^* : f'(0) = Bf(0)\}$. $\mathfrak{t}'_{A_B}$ is given by

$$\mathfrak{t}_{\widehat{A}_K}[f] = \int_0^\infty \left(|f'(x)| + Q(x)|f(x)|\right)^2 dx + M(0)|f(0)|^2. \quad (10)$$

If $Q = 0$, then $\kappa_-(A_{A_B}) = \kappa_-(B - M(0)) = \kappa_-(B)$

2.4 Direct sums of boundary triplets (see [5], [6], [7])

Let S_n be a densely defined symmetric operator in a Hilbert space $\mathfrak{H}_n$ with $\mathrm{n}_+(S_n) = \mathrm{n}_-(S_n) \le \infty$, $n \in \mathbb{N}$. Consider the operator $A := \bigoplus_{n=1}^{\infty} S_n$ acting in $\mathfrak{H} := \bigoplus_{n=1}^{\infty} \mathfrak{H}_n$. Clearly, $A^* = \bigoplus_{n=1}^{\infty} S_n^*$.
Let $\mathcal{H} := \bigoplus_{n=1}^{\infty} \mathcal{H}_n$ be a Hilbert direct sum of $\mathcal{H}_n$. Define mappings Γ_0 and Γ_1 by setting $\Gamma_j := \bigoplus_{n=1}^{\infty} \Gamma_j^{(n)}$.
We assume that the operator $A = \bigoplus_{n=1}^{\infty} S_n$ has a regular real point, i.e., there exists an $\varepsilon > 0$ such that

$$(a - \varepsilon, a + \varepsilon) \subset \cap_{n=1}^{\infty} \widehat{\rho}(S_n). \tag{11}$$

Theorem 11

Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of symmetric operators satisfying (11). Let also $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ be a boundary triplet for S_n^* such that $(a-\varepsilon, a+\varepsilon) \subset \rho(S_{n0})$, and let $M_n(\cdot)$ be the corresponding Weyl function. Then:

(*i*) $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ forms a B-generalized boundary triplet for $A^* = \bigoplus_{n=1}^{\infty} S_n^*$ if and only if

$$C_3 := \sup_{n\in\mathbb{N}} \|M_n(a)\|_{\mathcal{H}_n} < \infty \quad \text{and} \quad C_4 := \sup_{n\in\mathbb{N}} \|M_n'(a)\|_{\mathcal{H}_n} < \infty, \tag{12}$$

where $M_n'(a) := (dM_n(z)/dz)|_{z=a}$.

(*ii*) $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ is an ordinary boundary triplet for $A^* = \bigoplus_{n=1}^{\infty} S_n^*$ if and only if, in addition to (12), the following condition is satisfied:

$$C_5 := \sup_{n\in\mathbb{N}} \|\big(M_n'(a)\big)^{-1}\|_{\mathcal{H}_n} < \infty. \tag{13}$$

Corollary 11

Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of sym. operators satisfying (11). Let also $\widetilde{\Pi}_n = \{\mathcal{H}_n, \widetilde{\Gamma}_0^{(n)}, \widetilde{\Gamma}_1^{(n)}\}$ be a boundary triplet for S_n^* such that $(a-\varepsilon, a+\varepsilon) \subset \rho(S_{n0})$, $S_{n0} = S_n^* \upharpoonright \ker(\widetilde{\Gamma}_0^{(n)})$, and $\widetilde{M}_n(\cdot)$ the corresp. Weyl function. Assume also that for some operators R_n such that $R_n, R_n^{-1} \in [\mathcal{H}_n]$, the following conditions are satisfied:

$$\begin{aligned} &\sup_n \|R_n^{-1}(\widetilde{M}_n'(a))(R_n^{-1})^*\|_{\mathcal{H}_n} < \infty \quad \text{and} \\ &\sup_n \|R_n^*(\widetilde{M}_n'(a))^{-1}R_n\|_{\mathcal{H}_n} < \infty, \quad n \in \mathbb{N}. \end{aligned} \tag{14}$$

Then the direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ of boundary triplets

$$\begin{aligned} \Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\} \quad \text{with} \quad \Gamma_0^{(n)} := R_n\widetilde{\Gamma}_0^{(n)}, \\ \Gamma_1^{(n)} := (R_n^{-1})^*\big(\widetilde{\Gamma}_1^{(n)} - \widetilde{M}_n(a)\widetilde{\Gamma}_0^{(n)}\big), \end{aligned} \tag{15}$$

forms a boundary triplet for $A^* = \bigoplus_{n=1}^{\infty} S_n^*$.

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Thank you!