Boundary triplets approach to the extension theory and abstract Weyl function

Mark Malamud

Institute of Appl. Math. and Mech.

1.1 Boundary Triplets and Parameterization of Proper Extensions

Let A be a densely defined closed symmetric operator in a separable Hilbert space $\mathfrak{H}$ with equal deficiency indices $\mathrm{n}_\pm(A) = \dim \mathfrak{N}_{\pm i} \leq \infty$, where $\mathfrak{N}_z := \ker(A^* - z)$ is the defect subspace. Let $\mathfrak{H}_+$ is $\mathrm{dom}\, A^*$ equipped with the norm of the graph defined by equality $\|f\|_+^2 = \|f\|^2 + \|A^* f\|^2$.

Defintion 1

A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is called a boundary triplet for the adjoint operator A^* of A if $\mathcal{H}$ is an auxiliary Hilbert space and $\Gamma_0, \Gamma_1 : \ \mathrm{dom}\,(A^*) \to \mathcal{H}$ are linear mappings such that
(i) the following abstract second Green identity holds

$$(A^* f, g) - (f, A^* g) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \quad f, g \in \mathrm{dom}\,(A^*); \tag{1}$$

(ii) the mapping $\Gamma := (\Gamma_0, \Gamma_1)^\top : \mathrm{dom}\,(A^*) \to \mathcal{H} \oplus \mathcal{H}$ is surjective.

Lemma 1

Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a bound. tripl. of the op. A^*. Then
(*i*) $\Gamma_j \in \mathcal{B}(\mathfrak{H}_+, \mathcal{H})$, $j \in \{0, 1\}$ and $\ker \Gamma = \mathrm{dom}\, A =: \mathfrak{H}_+^0$.
(*ii*) Map. $\widetilde{\Gamma} : \mathfrak{H}_+/\mathfrak{H}_+^0 \to \mathcal{H} \oplus \mathcal{H}$ defines a topolog. isomorph.

Definition 2

(*i*) A closed extension $\widetilde{A}$ of A is called a proper extension, if $A \subsetneq \widetilde{A} \subsetneq A^*$. The set of all proper extensions of A completed by the (non-proper) extensions A and A^* is denoted by Ext_A.
(*ii*) Two proper extensions A', A'', of A are called disjoint if $\mathrm{dom}\,(A') \cap \mathrm{dom}\,(A'') = \mathrm{dom}\,(A)$ and transversal if in addition $\mathrm{dom}\,(A') + \mathrm{dom}\,(A'') = \mathrm{dom}\,(A^*)$.

The set $\widetilde{\mathcal{C}}(\mathcal{H})$ of closed linear relations in $\mathcal{H}$ is the set of closed linear subspaces of $\mathcal{H} \oplus \mathcal{H}$. Recall that $\mathrm{dom}\,(\Theta) = \big\{f : \{f, f'\} \in \Theta\big\}$, $\mathrm{ran}\,(\Theta) = \big\{f' : \{f, f'\} \in \Theta\big\}$, and $\mathrm{mul}\,(\Theta) = \big\{f' : \{0, f'\} \in \Theta\big\}$ are the domain, the range, and the multivalued part of Θ. A closed linear operator A in $\mathcal{H}$ is identified with its graph $\mathrm{gr}\,(A)$, so that the set $\mathcal{C}(\mathcal{H})$ of closed linear operators in $\mathcal{H}$ is viewed as a subset of $\widetilde{\mathcal{C}}(\mathcal{H})$. In particular, a linear relation Θ is an operator if and only if $\mathrm{mul}\,(\Theta)$ is trivial. Note that the adjoint relation $\Theta^* \in \widetilde{\mathcal{C}}(\mathcal{H})$ of $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$ is defined by

$$\Theta^* = \left\{ \begin{pmatrix} h \\ h' \end{pmatrix} : (f', h)_{\mathcal{H}} = (f, h')_{\mathcal{H}} \text{ for all } \begin{pmatrix} f \\ f' \end{pmatrix} \in \Theta \right\}.$$

A linear relation Θ is said to be symmetric if $\Theta \subset \Theta^*$ and self-adjoint if $\Theta = \Theta^*$.
For a symmetric linear relation $\Theta \subseteq \Theta^*$ in $\mathcal{H}$ the multivalued part $\mathrm{mul}\,(\Theta)$ is the orthogonal complement of $\mathrm{dom}\,(\Theta)$ in $\mathcal{H}$.

Proposition 1

The map $\Gamma : \mathfrak{H}_+ \to \mathcal{H} \oplus \mathcal{H}$ defines a bijective corres-ce between the set of Ext_A and the set $\widetilde{\mathcal{C}}(\mathcal{H})$ of closed linear relat-s in $\mathcal{H}$.

$$\mathrm{Ext}_A \ni \widetilde{A} \mapsto \Theta := \Gamma(\mathrm{dom}\,\widetilde{A}) = \{(\Gamma_0 f \quad \Gamma_1 f)^T : f \in \mathrm{dom}\,\widetilde{A}\} \in \widetilde{\mathcal{C}}(\mathcal{H}),$$

(we will write $A_\Theta := \widetilde{A}$). Moreover, the following holds:
(*i*) $(A_\Theta)^* = A_{\Theta^*}$;
(*ii*) $A_{\Theta_1} \subseteq A_{\Theta_2} \Leftrightarrow \Theta_1 \subseteq \Theta_2$;
(*iii*) A_Θ is symmetric $(A_\Theta \subseteq (A_\Theta)^*) \Leftrightarrow \Theta$ is symmetric and $n_\pm(A_\Theta) = n_\pm(\Theta)$. In particular, $A_\Theta = (A_\Theta)^* \Leftrightarrow \Theta = \Theta^*$;
(*iv*) A_{Θ_1} and A_{Θ_2} are disjoint $\Leftrightarrow \Theta_1 \cap \Theta_2 = \{0\}$;
(*v*) A_{Θ_1} and A_{Θ_2} are transversal $\Leftrightarrow \Theta_1 \dot{+} \Theta_2 = \mathcal{H} \oplus \mathcal{H}$;
(*vi*) A_Θ and $\widetilde{A}_0$ disjoint (transversal) $\Leftrightarrow \Theta = \mathrm{gr}\, B,\ B \in \mathcal{C}(\mathcal{H})$ $(B \in \mathcal{B}(\mathcal{H}))$. In this case

$$\mathrm{dom}\,\widetilde{A} = \ker(\Gamma_1 - B\Gamma_0), \quad B \in \mathcal{C}(\mathcal{H}),$$

and B is called a boundary operator for the extension $\widetilde{A} = A_B$.

Definition 3

Let A be a symmetric oper-r in $\mathfrak{H}$, $\tilde{A} = \tilde{A}^* \in \mathrm{Ext}_A$ and $\mathcal{H}$ a Hilbert space, $\dim \mathcal{H} = n_\pm(A)$. The op-r valued function $\gamma : \rho(\widetilde{A}) \to \mathcal{B}(\mathcal{H}, \mathfrak{H})$ is called the γ-field of A, corresponding to ext-n $\tilde{A}$, if:
(i) $\gamma(\lambda)$ isomorphically maps $\mathcal{H}$ to $\mathfrak{N}_\lambda$ for all $\lambda \in \rho(\widetilde{A})$;
(ii) the identity is valid:

$$\gamma(\lambda) = U_{\zeta,\lambda}\gamma(\zeta) := [I + (\lambda - \zeta)(\widetilde{A} - \lambda)^{-1}]\gamma(\zeta), \quad \lambda, \zeta \in \rho(\widetilde{A}). \tag{2}$$

Lemma 2

Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for the operator A^*, $A_0 := A^* \upharpoonright \ker \Gamma_0$. Then:
(i) For each $\lambda \in \rho(A_0)$ a direct sum decomposition

$$\mathrm{dom}\, A^* = \mathrm{dom}\, A_0 \dotplus \mathfrak{N}_\lambda, \qquad \lambda \in \rho(A_0) \tag{3}$$

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(*ii*) The operator valued function

$$\gamma(\lambda) := (\Gamma_0 \upharpoonright \mathfrak{N}_\lambda)^{-1}, \qquad \lambda \in \rho(A_0) \tag{4}$$

is well defined and holom-c in $\rho(A_0)$ with values in $\mathcal{B}(\mathcal{H}, \mathfrak{N}_\lambda)$.
(*iii*) $\gamma(\lambda)$ is the γ–field of the op. A, corr-ing to extension A_0.
(*iv*) The following identity holds

$$\gamma(\overline{\lambda})^* = \Gamma_1(A_0 - \lambda)^{-1}, \quad \lambda \in \rho(A_0). \tag{5}$$

Definition 4

Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. The operator valued function $M(\cdot)$ defined by

$$M(\lambda)\Gamma_0 f_\lambda = \Gamma_1 f_\lambda, \qquad f_\lambda \in \mathfrak{N}_\lambda, \qquad \lambda \in \rho(A_0), \tag{6}$$

is called the Weyl function of the operator A, corresponding to the boundary triplet Π.

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Theorem 1

Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for the operator A^*, $M(\cdot)$ the corresponding Weyl function. Then:

(i) $M(\cdot)$ is well defined and holomorphic in $\rho(A_0)$ as an operator valued function in $\mathcal{B}(\mathcal{H})$;

(ii) for all $\lambda, \zeta \in \rho(A_0)$ the following identity is valid

$$M(\lambda) - M(\zeta)^* = (\lambda - \overline{\zeta})\gamma(\zeta)^*\gamma(\lambda), \quad \lambda, \zeta \in \rho(A_0); \tag{7}$$

(iii) $M(\cdot)$ is $R[\mathcal{H}]$-function and satisfies the condition

$$0 \in \rho(\mathrm{Im}\, M(\lambda)), \qquad \lambda \in \mathbb{C}_+ \cup \mathbb{C}_-. \tag{8}$$

Proposition 2

Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for the operator A^*, $M(\cdot)$ the corresponding Weyl function. Then:

(*i*) The Weyl function $M(\cdot)$ admits an integral representation

$$M(\lambda) = C_0 + \int_{\mathbb{R}} \left(\frac{1}{t-\lambda} - \frac{t}{1+t^2} \right) d\Sigma(t). \tag{9}$$

where $C_0 = {C_0}^* \in \mathcal{B}(\mathcal{H})$ and $\Sigma(\cdot) = \Sigma^*(\cdot)$ is a non-decreasing operator valued function with values in $\mathcal{B}(\mathcal{H})$ satisfying

$$\int_{\mathbb{R}} (1+t^2)^{-1} d\Sigma(t) \in \mathcal{B}(\mathcal{H}) \quad \text{and} \quad \int_{\mathbb{R}} d(\Sigma(t)h, h) = +\infty, \quad h \in \mathcal{H}\setminus\{0\}.$$

(*ii*) If $M(\cdot)$ admits a holomorphic continuation through the interval (α, β), then $M(\lambda)$ is strictly increasing in (α, β), i.e.

$$M(\lambda_1) \leq M(\lambda_2) \quad \text{and} \quad 0 \in \rho(M(\lambda_2) - M(\lambda_1)) \quad \text{for} \quad \alpha < \lambda_1 < \lambda_2 < \beta. \tag{10}$$

The spectrum of the proper extension A_Θ of the operator A can be described in terms of the Weyl function and the boundary relation Θ.

Theorem 2

Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*, $M(\cdot)$ the corresponding Weyl function, $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$ and $\lambda \in \rho(A_0)$. Then the following equivalences are true:

$$\lambda \in \rho(A_\Theta) \Longleftrightarrow 0 \in \rho(\Theta - M(\lambda)); \tag{11}$$

$$\lambda \in \sigma_i(A_\Theta) \Longleftrightarrow 0 \in \sigma_i(\Theta - M(\lambda)), \qquad i \in \{p, c, r\}; \tag{12}$$

At the same time we have the equalities

$$\ker(A_\Theta - \lambda) = \gamma(\lambda) \ker(\Theta - M(\lambda)). \tag{13}$$

Proposition 3

Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ and $\widetilde{\Pi} = \{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ be the boundary triplets for A^*, which are related by J-unitary operator $X = (X_{jk})_{j,k=1}^2 \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. Then the Weyl functions $M(z)$ and $\widetilde{M}(z)$, corresponding to the boundary triplets Π and $\widetilde{\Pi}$ are related as

$$\widetilde{M}(z) = (X_{11}M(z) + X_{12})(X_{21}M(z) + X_{22})^{-1}, \qquad (14)$$

In particular, if $\ker \widetilde{\Gamma}_0 = \ker \Gamma_0 = \mathrm{dom}\,(A_0)$ and boundary triplets Π and $\widetilde{\Pi}$ are related by

$$\widetilde{\Gamma}_0 = Z^{-1}\Gamma_0, \qquad \widetilde{\Gamma}_1 = Z^*(\Gamma_1 + K\Gamma_0), \qquad (15)$$

then the Weyl functions $M(\cdot)$ and $\widetilde{M}(\cdot)$ are related by

$$\widetilde{M}(z) = Z^*M(z)Z + K \qquad (16)$$

where $K = K^* \in \mathcal{B}(\mathcal{H})$, $Z \in \mathcal{B}(\mathcal{H})$ is boundedly invertible.

Theorem 3

Suppose that $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ and $\widetilde{\Pi} = \{\widetilde{\mathcal{H}}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ are two boundary triplets for A^*, V — isometry from $\widetilde{\mathcal{H}}$ to $\mathcal{H}$ and $\widetilde{V} = V \oplus V$. Then there exists bounded $J_{\mathcal{H}}$-unitary operator $X = (X_{i,j})_{i,j=1}^2$ in $\mathcal{H} \oplus \mathcal{H}$, such that

$$\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ \Gamma_0 \end{pmatrix} = \begin{pmatrix} V\widetilde{\Gamma}_1 \\ V\widetilde{\Gamma}_0 \end{pmatrix}. \tag{17}$$

1.3 Example 1. First-order differentiation operator on a finite interval

Let $P = P_{\min}$ and $P_{\max}$ be the minimal and maximal operators, generated in $L^2(0,1)$ by the dif-al expression $D := -i\frac{d}{dx}$. Then:

$$\operatorname{dom} P = H_0^1[0,1] = \{f \in H^1[0,1] : f(0) = f(1) = 0\}.$$

Moreover, $P_{\max} = P^*$ and $\operatorname{dom} P^* = H^1[0,1]$. From the equation $(P^* - z)f = 0$, we find the defect subspace of the operator P:

$$\mathfrak{N}_z = \ker(P^* - z) = \operatorname{span}\{e^{izx}\}, \quad z \in \mathbb{C},$$

hence, the deficiency indices of P are $n_\pm(P) = 1$.
The boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ of the operator P^* can be chosen as

$$\sqrt{2}\Gamma_0 f = f(0) + f(1), \quad \sqrt{2}\Gamma_1 f = i(f(0) - f(1)), \quad \mathcal{H} = \mathbb{C}. \quad (18)$$

Let $f_z = e^{izx}$, we find

$$\Gamma_0 f_z = \sqrt{2} e^{iz/2} \cos(z/2), \quad \Gamma_1 f_z = \sqrt{2} e^{iz/2} \sin(z/2),$$

hence, the corresponding Weyl function $M(\cdot)$ is

$$M(z) = \Gamma_1 (\Gamma_0 \upharpoonright \mathfrak{N}_z)^{-1} = \mathrm{tg}(z/2), \qquad z \in \mathbb{C}_+ \cup \mathbb{C}_-.$$

Clearly, $M(\cdot)$ is a meromorphic function. Let $P_j = P^* \upharpoonright \ker \Gamma_j$, $j \in \{0, 1\}$, we see that the singularities $M(\cdot)$ coincide with the spectrum of the operator P_0, $\sigma(P_0) = \{\pi + 2\pi k\}_{k \in \mathbb{Z}}$, and zeros coincide with the spectrum of the operator P_1, $\sigma(P_1) = \{2\pi k\}_{k \in \mathbb{Z}}$.

Example 2. Differentiation operator on the line.

Let P be the direct sum of minimal operators associated with the differential expression $D := -i\frac{d}{dx}$ on the half-lines,

$$\operatorname{dom} P = \{f \in H^1(\mathbb{R}) : f(0) = 0\} = H^1_0(\mathbb{R}_-) \oplus H^1_0(\mathbb{R}_+).$$

It is easily seen that the operator P can be represented as

$$P = P_- \oplus P_+, \tag{19}$$

where $P_\pm := P_{\pm,\min} : f \mapsto -i\frac{df}{dx}$ are the minimal operators in $L^2(\mathbb{R}_\pm)$. Moreover, $P_{\max} = P^*$. It's clear that $P^* = P^*_- \oplus P^*_+$ and $\operatorname{dom} P^* = H^1(\mathbb{R}_-) \oplus H^1(\mathbb{R}_+)$. We find the defect subspace of P

$$\mathfrak{N}_{\pm z} = \ker(P^* - z) = \operatorname{span}\{e^{izx}\chi_{\mathbb{R}_\pm}(x)\}, \qquad z \in \mathbb{C}_\pm.$$

It is easy to see that $(n_-(P_-), n_+(P_-)) = (1, 0)$ and $(n_-(P_+), n_+(P_+)) = (0, 1)$. Therefore, by (19) $n_\pm(P) = 1$.

The boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ of the operator P^* can be chosen in the form

$$\sqrt{2}\Gamma_0 f := f(+0) + f(-0), \quad \sqrt{2}\Gamma_1 f := i(f(+0) - f(-0)). \tag{20}$$

Let $f_z = e^{izx}\chi_{\mathbb{R}_\pm}(x)$, $z \in \mathbb{C}_\pm$, then the corresponding Weyl function $M(\cdot)$ has the form

$$M(z) = \Gamma_1(\Gamma_0 \upharpoonright \mathfrak{N}_z)^{-1} = \begin{cases} i, & z \in \mathbb{C}_+, \\ -i, & z \in \mathbb{C}_-. \end{cases} \tag{21}$$

Hence the spectral function $\sigma_M(t)$ is determined by applying the Stieltjes inversion formula:

$$\sigma_M(t) = \frac{1}{\pi}\int_0^t \operatorname{Im} M(x + i0)dx = \frac{1}{\pi}\int_0^t \operatorname{Im} i dx = \frac{1}{\pi}\int_0^t dx = \frac{t}{\pi}.$$

Example 3. Sturm–Liouville operator on a finite interval.

Let $A := A_{\min}$ and $A_{\max}$ be the minimal and maximal operators, respectively, generated in $L^2(0,1)$ by the dif. expression $\mathcal{A} = -\frac{d^2}{dx^2} + q$ with a real potential $q = \bar{q} \in L^2(0,1)$. In this case

$$A_{\max} = A^* \quad \text{and} \quad \operatorname{dom} A^* = H^2[0,1], \tag{22}$$

and

$$\operatorname{dom} A = H_0^2[0,1] = \{f \in H^2[0,1] : \ f(0) = f(1) = f'(0) = f'(1) = 0\}. \tag{23}$$

Since $q = \overline{q}$, the operator A is symmetric. Let $c(\cdot, z)$, $s(\cdot, z)$ be the solutions of eq-on $\mathcal{A}f = zf$ satisfying the initial cond-s

$$s(0,z) = c'(0,z) = 0, \qquad\qquad s'(0,z) = c(0,z) = 1. \tag{24}$$

Then the defect subspace $\mathfrak{N}_z$ is $\mathfrak{N}_z = \operatorname{span}\{c(\cdot, z), s(\cdot, z)\}$. Hence, $n_\pm(A) = 2$.

The set $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ with

$$\mathcal{H} = \mathbb{C}^2, \quad \Gamma_0 f = \begin{pmatrix} f(0) \\ f(1) \end{pmatrix}, \quad \Gamma_1 f = \begin{pmatrix} f'(0) \\ -f'(1) \end{pmatrix}. \tag{25}$$

forms a boundary triplet for the operator A^*.
The corresponding Weyl function $M(\cdot)$ and γ-field are given by

$$M(z) = -\frac{1}{s(1,z)} \begin{pmatrix} c(1,z) & -1 \\ -1 & s'(1,z) \end{pmatrix}. \tag{26}$$

$$\gamma(z) = \frac{1}{s(1,z)} \begin{pmatrix} c(\cdot,z) & s(\cdot,z) \end{pmatrix} \begin{pmatrix} c(1,z) & -1 \\ -1 & s'(1,z) \end{pmatrix}. \tag{27}$$

Example 4. Sturm–Liouville operator on the semiaxis

Let $A := A_{\min}$ and $A_{\max}$ be the minimal and maximal operators generated in $L^2(\mathbb{R}_+)$ by the differential expression $\mathcal{A} = -\frac{d^2}{dx^2} + q$ with a potential $q = \bar{q} \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$. Using the embedding theorem, one proves the inclusions

$$\mathrm{dom}\, A \subset \{f \in H^2_{\mathrm{loc}}(\mathbb{R}_+) : \ f(0) = f'(0) = 0\} \text{ and } \quad \mathrm{dom}\, A_{\max} \subset H^2_{\mathrm{loc}}(\mathbb{R}_+)$$

and equality $A_{\max} = A^*$. If $q = \bar{q} \in L^\infty(\mathbb{R}_+)$, then

$$\mathrm{dom}\, A = H^2_0(\mathbb{R}_+) \qquad \text{and} \qquad \mathrm{dom}\, A_{\max} = H^2(\mathbb{R}_+).$$

Now let the potential $q \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$ be semibounded from below. In that case $n_\pm(A) = 1$ (the limit point case at infinity). Hence $\lim_{b\to\infty} f(b) = \lim_{b\to\infty} f'(b) = 0$ for any $f \in \mathrm{dom}\, A_{\max}$ and the Green's formula for the operator A^* takes the form

$$\int_a^\infty (\mathcal{A}f)(x)\overline{g(x)}dx - \int_0^\infty f(x)\overline{(\mathcal{A}g)(x)}dx = -[f,g]_0,$$

where $[f,g]_x := f(x)\overline{g'(x)} - f'(x)\overline{g(x)}$.

Therefore, the boundary triplet $\Pi^\infty = \{\mathcal{H}, \Gamma_0^\infty, \Gamma_1^\infty\}$ for the operator A^* can be chosen in the form

$$\mathcal{H} = \mathbb{C}, \quad \Gamma_0^\infty f = f(0), \quad \Gamma_1^\infty f = f'(0). \tag{28}$$

Let $u_1, u_2 \in \operatorname{dom} A^*$ be smooth functions with compact support in $[0, b]$, $b < \infty$, and satisfying the conditions

$$u_1(0) = 1, \quad u_2(0) = 0, \qquad u_1'(0) = 0, \quad u_2'(0) = 1. \tag{29}$$

Then $[f, u_1]_\infty = [f, u_2]_\infty = 0$ and

$$f(0) = [f, u_2]_0, \quad f'(0) = -[f, u_1]_0.$$

Therefore, the mapping $\Gamma^\infty = \{\Gamma_0^\infty, \Gamma_1^\infty\} : \operatorname{dom} A^* \to \mathbb{C} \times \mathbb{C}$ is surjective.
Let $c(x, z)$, $s(x, z)$ be the solutions of the equation $\mathcal{A}[f] = zf$, satisfying

$$c(0, z) = 1, \quad c'(0, z) = 0,$$
$$s(0, z) = 0, \quad s'(0, z) = 1.$$

Since $n_{\pm}(A) = 1$, for each $z \in \mathbb{C}\setminus\mathbb{R}$ there exists the unique (Weyl) coefficient $m(z)$, such that

$$f_z(x) := c(x,z) + m(z)s(x,z) \in L^2(\mathbb{R}_+).$$

The function $f_z(\cdot)$ is called the Weyl solution of equation $\ell[f] - zf = 0$. Clearly, $\Gamma_0^{\infty} f_z = 1$, $\Gamma_1^{\infty} f_z = m(z)$, and the Weyl function $M_{\infty}(\cdot)$ corresponding to the boundary triplet (28) is

$$M_{\infty}(z) = (\Gamma_1^{\infty} f_z)(\Gamma_0^{\infty} f_z)^{-1} = m(z).$$

Thus, it coincides with the classical Weyl coefficient $m(\cdot)$.

Example 5. Ordinary dif. op-s of order $2n$ on a finite interval

Let $A := A_{\min}$ be the min. op-r generated in $L^2(0,1))$ by dif. exp-n

$$(\mathcal{A}[f])(x) = \sum_{k=1}^{n} (-1)^k (p_{n-k}(x) f^{(k)}(x))^{(k)} + p_n(x) f(x),$$

where $p_0^{-1}, p_1, \ldots, p_n$ is a real, measurable and summable functions on $(0,1)$. Then the operator A has deficiency indices $(2n, 2n)$. Let the quasi-derivatives $f^{[k]}$ be defined by the equalities

$$f^{[k]}(x) = f^{(k)}(x), \quad k \in \{0, \ldots, n-1\}, \qquad f^{[n]}(x) = p_0(x) f^{(n)}(x),$$

$$f^{[n+k]}(x) = p_k(x) f^{(n-k)}(x) - \frac{d}{dx} f^{[n+k-1]}(x), \quad k \in \{1, \ldots, n\}.$$

The domain of the operator $A_{\max} =: A^*$ is

$$\operatorname{dom} A^* = \{f \in L^2(0,1) : \mathcal{A}[f] \in L^2(0,1)\},$$

and the domain of the minimal operator A is given as follows

$$\mathrm{dom}\, A = \{f \in \mathrm{dom}\, A^* : \ f^{[k]}(0) = f^{[k]}(1) = 0,\ k \in \{0, 1, \dots, 2n-1\}\}.$$

For any pair of func. $f, g \in \mathrm{dom}\, A^*$ the Lagrange identity holds:

$$\mathcal{A}[f]\overline{g}(x) - f\mathcal{A}[\overline{g}](x) = \frac{d}{dx}[f, g]_x, \tag{30}$$

where

$$[f, g]_x = \sum_{k=1}^{n} \left(f^{[k-1]}(x)\overline{g}^{[2n-k]}(x) - f^{[2n-k]}(x)\overline{g}^{[k-1]}(x) \right). \tag{31}$$

The totality $\Pi = \{\mathbb{C}^{2n}, \Gamma_0, \Gamma_1\}$ where

$$\Gamma_0 f = \begin{pmatrix} \widehat{f}_0(0) \\ \widehat{f}_0(1) \end{pmatrix}, \quad \Gamma_1 f = \begin{pmatrix} \widehat{f}_1(0) \\ -\widehat{f}_1(1) \end{pmatrix}, \\ \widehat{f}_0 := \left(f, \dots, f^{(n-1)}\right)^\top, \quad \widehat{f}_1 := \left(f^{[2n-1]}, \dots, f^{[n]}\right)^\top, \tag{32}$$

forms a boundary triplet for A^*.

The corresponding Weyl function $M(z)$ is

$$M(z) = Y_1(z)(Y_0(z))^{-1}, \quad Y_j(z) = \Gamma_j(V(t, z)), \quad j \in \{0, 1\},$$

where $V(t,z) = (v_1(t,z), ..., v_{2n}(t,z))^\top$ is a fundamental system of matrix solutions of equation $\mathcal{A}[f] = zf$ satisfying the initial conditions $v_j^{[k-1]} = \delta_{jk} I_n$, $j, \in \{1, ..., 2n\}$.

Krein formula for the resolvents

Theorem 4

Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*, $M(\cdot)$ the corresponding Weyl function, $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$, and A_Θ the corresponding proper extension of the operator A. Then:
(i) The formula

$$\mathrm{dom}\,(A_\Theta) = \{f \in \mathrm{dom}\,A^* : \Gamma f \in \Theta\}, \quad \Theta := \Gamma(\mathrm{dom}\,A_\Theta) \qquad (34)$$

establish a bijective correspondence between the set of all proper extensions A_Θ of the operator A and the set of closed linear relations $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H}) \setminus \{0\}$;
(ii) for each $\Theta \in \widetilde{\mathcal{C}}(\mathcal{H})$, such that $\rho(A_\Theta) \neq \emptyset$, the following Krein type formula holds

$$(A_\Theta - z)^{-1} = (A_0 - z)^{-1} + \gamma(z)(\Theta - M(z))^{-1}\gamma(\overline{z})^*, \quad z \in \rho(A_0) \cap \rho(A_\Theta); \qquad (35)$$

(iii) equality (35) establish a second bijective correspondence between the set of proper extensions A_Θ of the operator A, for which $\rho(A_\Theta) \neq \emptyset$, and the set of closed linear relations Θ in $\widetilde{\mathcal{C}}(\mathcal{H})$ with $\{z : 0 \in \rho(\Theta - M(z))\} \neq \emptyset$.

Corollary 1

If the linear relation Θ is a graph of the operator $B \in \mathcal{C}(\mathcal{H})$, then formula (35) has the form

$$(A_B - z)^{-1} = (A_0 - z)^{-1} + \gamma(z)(B - M(z))^{-1}\gamma(\overline{z})^*; \qquad (36)$$

and for each $g \in \mathfrak{H}$ vector–function $f = (A_B - z)^{-1}g$ is a solution of the boundary value problem

$$(A^* - z)f = g, \quad \Gamma_1 f = B\Gamma_0 f, \quad z \in \rho(\widetilde{A}_B). \qquad (37)$$

Let $\mathfrak{S}(\mathfrak{H})$ be a two-sided ideal in the algebra $\mathcal{B}(\mathfrak{H})$.

Theorem 5 (On resolvent comparability)

Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be the boundary triplet for the operator A^*, $\Theta_1, \Theta_2 \in \widetilde{\mathcal{C}}(\mathcal{H})$, $z \in \rho(A_{\Theta_1}) \cap \rho(A_{\Theta_2}) \cap \rho(A_0)$. Then for every symmetric normed ideal $\mathfrak{S}$ in $\mathcal{C}(\mathcal{H})$ and for each $\zeta \in \rho(\Theta_1) \cap \rho(\Theta_2)$ the following equivalence is valid:

$$(A_{\Theta_1} - z)^{-1} - (A_{\Theta_2} - z)^{-1} \in \mathfrak{S}(\mathfrak{H}) \iff (\Theta_1 - \zeta)^{-1} - (\Theta_2 - \zeta)^{-1} \in \mathfrak{S}(\mathcal{H}). \tag{38}$$

Emphasize, that all objects in formula (35) are expressed by means of boundary mappings Γ_0 and Γ_1, and alongside (34) it gives the second parameterization of the set Ext_A. This fact makes formula (35) a power tool in the theory and applications of the boundary triplets' technique to different analytical problems, in particular, to boundary value problems.

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Thank you for your attention!