Dispersion decay for Schrödinger and Klein-Gordon equations

We are concerned with the Schrödinger equation

$$i\dot{\psi}(x,t) = H\psi(x,t), \quad H := -\Delta + V(x), \quad x \in \mathbb{R}^n, \ \ t \in \mathbb{R}$$

and the Klein–Gordon equation

$$\ddot{\psi}(x,t) = -(H + m^2)\psi(x,t), \quad m > 0, \quad x \in \mathbb{R}^n, \ \ t \in \mathbb{R}$$

with real potential V. In vector form Klein–Gordon equation reads

$$i\dot{\Psi}(t) = \mathcal{H}\Psi(t), \quad \Psi(t) = \begin{pmatrix} \psi(t) \\ \dot{\psi}(t) \end{pmatrix}, \quad \mathcal{H} = i\begin{pmatrix} 0 & 1 \\ -H - m^2 & 0 \end{pmatrix}$$

The weighted spaces $L^p_\sigma = L^p_\sigma(\mathbb{R}^n)$, $\sigma \in \mathbb{R}$, $1 \le p \le \infty$, with the norm

$$\|\psi\|_{L^p_\sigma} = \|\langle x\rangle^\sigma \psi\|_{L^p(\mathbb{R}^n)} < \infty, \quad \langle x\rangle = (1 + |x|^2)^{\frac{1}{2}}$$

The case $\sigma = 0$ corresponds to the usual L^p spaces without weight.

1 1D Schrödinger equation

$$i\dot{\psi}(x,t) = H\psi(x,t), \quad H := -\frac{\partial^2}{\partial x^2} + V(x), \quad (x,t) \in \mathbb{R}^2, \quad V \in L_1^1$$

$\Sigma_c = [0, \infty)$–purely absolutely continuous spectrum.

Definition 0 is a resonance if $\exists$ a nonzero solution $\psi \in L^\infty$ to $H\psi = 0$.

Theorem 1.1 *Let $V \in L_1^1$. Then*

$$\|e^{-itH} P_c\|_{L^1 \to L^\infty} = \mathcal{O}(t^{-1/2}), \quad t \to \infty \tag{1.1}$$

Here $P_c = P_c(H)$ is the orthogonal projection in L^2 onto the continuous spectrum of H.

$$\|K(t)\|_{L^1 \to L^\infty} = \sup_{\|f\|_{L^1}=1, \|g\|_{L^1}=1} \langle f, K(t)g \rangle = \sup_{x,y} |K(t,x,y)|$$

$$V = 0: \quad e^{-itH}(x,y) = e^{-\frac{|x-y|^2}{4it}} / \sqrt{4\pi it}$$

Hence

$$\|e^{-itH}\|_{L^1 \to L^\infty} = \sup_{x,y} |e^{-itH}(x,y)| \le Ct^{-1/2}, \ t \ge 1$$

The dispersion decay for $V \neq 0$ has been established

in [W] for $V \in L^1_\gamma$ with $\gamma > 3/2$ in the non-resonant case and $\gamma > 5/2$ in the resonant case.
in [GS] for $V \in L^1_1$ in the non-resonant case and $V \in L^1_2$ in the resonant case
in [EKMT] for $V \in L^1_1$ in both cases.

[W] R. Weder, $L^p - L^{\dot{p}}$ estimates for the Schrödinger equation on the line and inverse scattering for the nonlinear Schrödinger equation with a potential, *J. Funct. Anal.* **170** (2000), 37–68.

[GS] M. Goldberg, W. Schlag, Dispersive estimates for Schrödinger operators in dimensions one and three, *Comm. Math. Phys.* **251** (2004), 157-178.

[EKMN] I.Egorova, E.Kopylova, V.A.Marchenko, G. Teschl, Dispersion estimates for one-dimensional Schrödinger and Klein-Gordon equations revisited, *Russian Mathematical Surveys*, **71** (2016), no. 3, 391-415.

The decay (1.1) implies the following decay in weighted L^2-norms:

$$\|e^{-itH}P_c\|_{L^2_\sigma \to L^2_{-\sigma}} = \mathcal{O}(t^{-1/2}), \quad t \to \infty, \quad \sigma > 1/2 \tag{1.2}$$

Proof The norm of operator $e^{-itH}P_c : L^2_\sigma \to L^2_{-\sigma}$ is equivalent to the norm of $\langle x\rangle^{-\sigma} e^{-itH} P_c \langle y\rangle^{-\sigma} : L^2 \to L^2$

$$\Big(\int_{\mathbb{R}^2} |\langle x\rangle^{-\sigma}[e^{-itH}P_c](x,y)\langle y\rangle^{-\sigma}|^2 dxdy\Big)^{\frac{1}{2}} \le \sup_{x,y}[|e^{-itH}P_c](x,y)|\Big(\int_{\mathbb{R}^2}\langle x\rangle^{-2\sigma}\langle y\rangle^{-2\sigma} dxdy\Big)^{\frac{1}{2}}$$

$$\text{and} \quad \sup_{x,y}|[e^{-itH}P_c](x,y)| = \|e^{-itH}P_c\|_{L^1\to L^\infty} = \mathcal{O}(t^{-1/2})$$

Hence $e^{-itH}P_c$ is the Hilbert-Schmidt operator for $\sigma > 1/2$.

Theorem 1.2 *Let $V \in L^1_2(\mathbb{R})$. Then, in the non-resonant case,*

$$\|e^{-itH}P_c\|_{L^1_1 \to L^\infty_{-1}} = \mathcal{O}(t^{-3/2}), \quad t \to \infty \tag{1.3}$$

The dispersion decay (1.3) has been established

in [S] in the case $V \in L^1_4$
in [G] to the case $V \in L^1_3$.
in [M], [EKMT] for $V \in L^1_2$ (different approach)

[S] W. Schlag, Dispersive estimates for Schrödinger operators: a survey, in "Mathematical aspects of nonlinear dispersive equations", 255–285, *Ann. of Math. Stud.* **163**, Princeton Univ. Press, Princeton, NJ, 2007.

[G] M. Goldberg, Transport in the one dimensional Schrödinger equation, *Proc. Amer. Math. Soc.* **135** (2007), 3171-3179.

[M] H. Mizutani, Dispersive estimates and asymptotic expansions for Schrödinger equations in dimension one, *J. Math. Soc. Japan* **63** (2011), 239–261.

The decay (1.3) implies the following decay in weighted norms:

$$\|e^{-itH}P_c\|_{L^2_\sigma \to L^2_{-\sigma}} = \mathcal{O}(t^{-3/2}), \quad t \to \infty, \quad \sigma > 3/2 \tag{1.4}$$

The estimate of type (1.4) in the non-resonant case were obtained in [Mur] for more general (multi-dimensional) Schrödinger-type operators.

For $\sigma > 5/2$ and $|V(x)| \le C(1 + |x|)^{-\rho}$ with $\rho > 4$ in the 1D case.

[Mur] M. Murata, Asymptotic expansions in time for solutions of Schrödinger-type equations, *J. Funct. Anal.* **49** (1982), 10–56.

1.1 Continuity properties of the scattering matrix

Let $\mathcal{A}$ be the Banach algebra of Fourier transforms of integrable functions

$$\mathcal{A} = \left\{ f(k) : \ f(k) = \int e^{ikp} \hat{f}(p) dp, \ \hat{f}(\cdot) \in L^1 \right\}$$

with the norm $\|f\|_{\mathcal{A}} = \|\hat{f}\|_{L^1}$,

and let $\mathcal{A}_1$ be the corresponding unital Banach algebra

$$\mathcal{A}_1 = \left\{ f(k) : \ f(k) = c + \int e^{ikp} \hat{g}(p) dp, \ \hat{g}(\cdot) \in L^1, \ c \in \mathbb{C} \right\}$$

with the norm $\|f\|_{\mathcal{A}_1} = |c| + \|\hat{g}\|_{L^1}$.

Evidently, $\mathcal{A} \subset \mathcal{A}_1$.

If $f \in \mathcal{A}_1 \setminus \mathcal{A}$ and $f(k) \neq 0 \ \forall k \in \mathbb{R}$ then $f^{-1}(k) \in \mathcal{A}_1$ by the Wiener theorem.

We recall a few facts from scattering theory [DT] of the operator H. Under the assumption $V \in L^1_1$ there exist Jost solutions $f_\pm(x,k)$ of

$$H\psi = k^2\psi, \quad k \in \overline{\mathbb{C}_+},$$

normalized according to

$$f_\pm(x,k) \sim e^{\pm ikx}, \quad x \to \pm\infty$$

[DT] P. Deift and E. Trubowitz, Inverse scattering on the line, *Comm. Pure Appl. Math.* **32** (1979), 121–251.

These solutions are given by

$$f_{\pm}(x,k) = e^{\pm ikx} h_{\pm}(x,k), \quad h_{\pm}(x,k) = 1 \pm \int\limits_0^{\pm\infty} B_{\pm}(x,y) e^{\pm 2iky} dy$$

where $B_{\pm}(x,y)$ are real-valued and satisfy

$$\|B_{\pm}(x,\cdot)\|_{L^1} \leq e^{\gamma_{\pm}(x)} \gamma_{\pm}(x), \quad \|B'_{\pm}(x,\cdot)\|_{L^1} \leq \|V(x+\cdot)\|_{L^1} + 2e^{\gamma_{\pm}(x)} \gamma_{\pm}(x) \eta_{\pm}(x)$$

where

$$\gamma_{\pm}(x) = \int_x^{\pm\infty} (z-x)|V(z)|dz, \quad \eta_{\pm}(x) = \pm \int_x^{\pm\infty} |V(z)|dz$$

Hence,

$$h_{\pm}(x,\cdot) - 1 \in \mathcal{A}, \quad h'_{\pm}(x,\cdot) \in \mathcal{A}, \quad \forall x \in \mathbb{R} \tag{1.5}$$

Moreover,

$$\|h_\pm(x,\cdot)-1\|_{\mathcal{A}} \le C_\pm \quad \text{for} \quad \pm x \ge 0 \tag{1.6}$$

Indeed, in the "+" case one has

$$\gamma_+(x) = \int_x^\infty (z-x)|V(z)|dz \le \int_0^\infty z|V(z)|dz \le C_+, \quad x \ge 0$$

Wronskian : $W(\varphi(x,k),\psi(x,k)) = \varphi(x,k)\psi'(x.k) - \varphi'(x,k)\psi(x,k)$

$$W(k) = W(f_-(x,k), f_+(x,k)), \quad W_\pm(k) = W(f_\mp(x,k), f_\pm(x,-k))$$

The entries of the scattering matrix

$$T(k) = \frac{2ik}{W(k)}, \quad R_\pm(k) = \mp\frac{W_\pm(k)}{W(k)}$$

Theorem 1.3 *([EKMT]) If $V \in L^1_1$, then $T(k) - 1,\ R_\pm(k) \in \mathcal{A}$.*

Proof $W(k)$ can vanish only at $k = 0$ which is equivalent to the resonant case ([DT]). We consider the non-resonant case $W(0) \neq 0$ only.

Denote $h_\pm(k) := h_\pm(0,k),\ h'_\pm(k) := h'_\pm(0,k)$. Then

$$W(k) = 2ikh_+(k)h_-(k) + \tilde{W}(k)$$

$$\tilde{W}(k) := h_-(k)h'_+(k) - h'_-(k)h_+(k) \in \mathcal{A}$$

$$W_\pm(k) = h_\mp(k)h'_\pm(-k) - h_\pm(-k)h'_\mp(k) \in \mathcal{A}$$

$$f_\pm(x,k) = e^{\pm ikx}h_\pm(x,k), \quad f'_\pm(0,k) = \pm ikh_\pm(0,k) + h'_\pm(0,k)$$

$$W(k) = f_-(0,k)f'_+(0,k) - f'_-(0,k)f_+(0,k)$$

$$= h_-(k)(ikh_+(k) + h'_+(k) - h_+(k)(-ikh_-(k) + h'_-(k))$$

Denote

$$\nu(k) := \frac{1}{ik-1} = -\int_0^\infty e^{iky} e^{-y} dy$$

We have

$$\nu(k) \in \mathcal{A}, \quad \nu(k) W_\pm(k) \in \mathcal{A}, \quad \nu(k)\tilde{W}(k) \in \mathcal{A}, \quad ik\nu(k) \in \mathcal{A}_1$$

Hence,

$$\nu(k)W(k) = 2ik\nu(k)h_+(k)h_-(k) + \nu(k)\tilde{W}(k) \in \mathcal{A}_1$$

Further,

$$\nu(k)W(k) \to 2, \quad k \to \infty, \quad \text{then} \quad \nu(k)W(k) \in \mathcal{A}_1 \setminus \mathcal{A}$$

Moreover, $\nu(k)W(k) \neq 0$, $\forall k \in \mathbb{R}$, whence $(\nu(k)W(k))^{-1} \in \mathcal{A}_1$.

$$R_\pm(k) = \mp \frac{\nu(k)W_\pm(k)}{\nu(k)W(k)} \in \mathcal{A}, \quad T(k) = \frac{2ik\nu(k)}{\nu(k)W(k)} \in \mathcal{A}_1$$

Finally, $T(k) \to 1$ as $k \to \infty$. Then $T(k) - 1 \in \mathcal{A}$.

1.2 Proof of the theorem 1.1

We use the following representation (cf. [KK12])

$$
\begin{aligned}
e^{-itH}P_c &= \frac{1}{2\pi i}\int\limits_0^\infty e^{-it\omega}R(\omega+i0)-R(\omega-i0))\,d\omega \\
&= \frac{1}{\pi i}\int\limits_0^\infty e^{-itk^2}R(k^2+i0)-R(k^2-i0))\,kdk
\end{aligned}
$$

where $R(\omega)=(H-\omega)^{-1}$ is the resolvent of the operator H.
We express the kernel of the resolvent in terms of the Jost solutions [DT]
:

$$
R(k^2\pm i0,x,y)=\mp\frac{f_+(y,\pm k)f_-(x,\pm k)T(\pm k)}{2ik}
$$

for all $x\le y$ (and the positions of x,y reversed if $x>y$).

[KK12] A. Komech, E. Kopylova, Dispersion decay and scattering theory. John Willey & Sons, Hoboken, New Jersey, 2012.

In the case $x \le y$,

$$R(k^2+i0)-R(k^2-i0) = -\frac{f_+(y,k)f_-(x,k)T(k)}{2ik} - \frac{f_+(y,-k)f_-(x,-k)T(-k)}{2ik}$$

Hence,

$$\begin{aligned}
[e^{-itH}P_c](x,y) &= \frac{i}{\pi}\int\limits_{-\infty}^{\infty} e^{-itk^2}\,\frac{f_+(y,k)f_-(x,k)T(k)}{2ik}k\,dk \\
&= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-i(tk^2-|y-x|k)}h_+(y,k)h_-(x,k)T(k)dk \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(tk^2-|y-x|k)}(1+\psi(x,y,k))dk \qquad (1.7)
\end{aligned}$$

$$\begin{aligned}
\text{where} \quad \psi(x,y,k) &= h_+(y,k)h_-(x,k)T(k)-1 \quad \text{for} \quad x \le y \qquad (1.8) \\
\text{and} \quad \psi(x,y,k) &= \psi(y,x,k) \qquad \text{for} \quad x > y
\end{aligned}$$

Lemma 1 $\|\psi(x,y,\cdot)\|_{\mathcal{A}} \leq C, \quad x,y \in \mathbb{R}$

Proof We consider the three possibilities

$$(a)\ x \leq y \leq 0, \qquad (b)\ 0 \leq x \leq y, \qquad (c)\ x \leq 0 \leq y$$

In the case (c) the estimate follows immediately from (1.6) and Theorem 1.3. In the other two cases we use the scattering relations [DT]

$$T(k)h_\pm(x,k) = R_\mp(k)h_\mp(x,k)e^{\mp 2ikx} + h_\mp(x,-k)$$

to get the representation

$$\psi(k,x,y) = \begin{cases} h_-(x,k)\big(R_-(k)h_-(y,k)e^{-2iyk} + h_-(y,-k)\big) - 1, & x \leq y \leq 0 \\ h_+(y,k)\big(R_+(k)h_+(x,k)e^{2ixk} + h_+(x,-k)\big) - 1, & 0 \leq x \leq y \end{cases}$$

It remains to note that $\|g(k)e^{iks}\|_{\mathcal{A}} = \|g(k)\|_{\mathcal{A}}$, $\forall s \in \mathbb{R}$.

Lemma 2 $[e^{-itH}P_c](x,y) = \frac{1}{\sqrt{4\pi it}}\Big(e^{-\frac{|x-y|^2}{4it}} + \int\limits_{-\infty}^{\infty} e^{-\frac{(p+|x-y|)^2}{4it}}\hat{\psi}(x,y,p)dp\Big)$

Proof $[e^{-itH}P_c](x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(tk^2-|y-x|k)}(1+\psi(x,y,k))dk$ by (1.7)

The first part is easy to compute, we focus on the second part with ψ.

$$\frac{1}{2\pi}\lim_{k_0\to\infty}\int\limits_{-k_0}^{k_0}\int\limits_{-\infty}^{\infty} e^{-i(tk^2-|y-x|k-kp)}\hat{\psi}(x,y,p)dp\,dk$$

$$= \frac{1}{2\pi}\lim_{k_0\to\infty}\int\limits_{-\infty}^{\infty} e^{i\frac{(p+|y-x|)^2}{4t}}\hat{\psi}(x,y,p)\Bigg(\int\limits_{-k_0}^{k_0} e^{-i\frac{(2kt-|y-x|-p)^2}{4t}}dk\Bigg)dp$$

$$= \frac{1}{\sqrt{8\pi t}}\lim_{k_0\to\infty}\int\limits_{-\infty}^{\infty} e^{i\frac{(p+|y-x|)^2}{4t}}\hat{\psi}(x,y,p)\Big(\int\limits_{q_-}^{q_+} e^{-i\frac{\pi}{2}u^2}du\Big)dp$$

$$= \frac{1}{\sqrt{8\pi t}} \int_{-\infty}^{\infty} e^{i\frac{(p+|y-x|)^2}{4t}} \hat{\psi}(x,y,p) \lim_{k_0 \to \infty} (C(q_+) - iS(q_+) - C(q_-) + iS(q_-)) dp$$

where

$$q_\pm = \frac{\pm 2k_0 t - |x-y| - p}{\sqrt{2\pi t}},$$

and $C(z)$, $S(z)$ are the Fresnel integrals

$$C(z) = \int_0^z \cos(\frac{1}{2}\pi u^2) du, \quad S(z) = \int_0^z \sin(\frac{1}{2}\pi u^2) du$$

The Fresnel integrals are uniformly bounded and satisfy

$$C(\pm z) \to \pm\frac{1}{2}, \quad S(\pm z) \to \pm\frac{1}{2}, \quad z \to \infty$$

Hence, the claim follows. $(\frac{1}{\sqrt{8\pi t}}(1-i) = \frac{1}{\sqrt{4\pi i t}})$

Proof of Theorem 1.1

$$\begin{aligned}\|e^{-itH}P_c\|_{L^1\to L^\infty} &= \sup_{x,y}|[e^{-itH}P_c](x,y)| \le Ct^{-1/2}(1+\|\hat{\psi}(x,y,\cdot)\|_{L^1}) \\ &= Ct^{-1/2}(1+\|\psi(x,y,\cdot)\|_{\mathcal{A}}) \le Ct^{-1/2}, \quad t\ge 1\end{aligned}$$

due to Lemmas 1 and 2.