Lectures on Attractors of Hamilton Nonlinear PDEs

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1 Attractors of dynamical systems

1.0.1 Attractors of dynamical systems

Let $\mathscr{E}$ be a topological space,

Let $U(t) : \mathscr{E} \to \mathscr{E}$ be a semigroup of maps

Definition $\mathscr{A} \subset \mathscr{E}$ is the *point* attractor of the group $U(t)$ if

a) $U(t)Y \to \mathscr{A}$ as $t \to +\infty$ *for any* $Y \in \mathscr{E}$

b) This attraction fails for any proper open subset $\mathscr{A}' \subset \mathscr{A}$

1.1 Examples

1.1.1 Hamilton system: harmonic oscillator

$\ddot{x}(t) = -\omega^2 x(t)$, the elastic force

Hamiltonian $H(x,p) = \frac{p^2}{2} + \omega^2 \frac{x^2}{2}$

$\dot{x} = H_p$, $\dot{p} = -H_x$

solutions $x(t) = x_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t$

with initial data $x(0) = x_0, \quad \dot{x}(0) = v_0$

Lemma *Energy conservation*: $H(x(t), v(t)) = \text{const}$

Proof $\dot{H} = H_x \dot{x} + H_p \dot{p} = H_x H_p + H_p(-H_x) = 0$ □

Phase trajectories: *ellipces*, $\frac{p^2}{2} + \omega^2 \frac{x^2}{2} = C$

Attractor: $\mathscr{A} = \mathbb{R}^2$

1.1.2 Dissipative system: damped oscillator

$\ddot{x}(t) = -\omega^2 x(t) - \gamma\dot{x}(t), \qquad \gamma > 0$ (*friction coefficient*)

equations $\dot{x} = p = H_p$, $\dot{p} = -\omega^2 x - \gamma p = -H_x - \gamma p$

Energy dissipation:

$\dot{H} = H_x\dot{x} + H_p\dot{p} = H_x H_p + H_p(-H_x - \gamma p) = -\gamma p^2 < 0$

Phase trajectories – *spirals*:

$|x(t)| + |p(t)| \to 0, \quad t \to \infty$

Attractor: $\mathscr{A} = \{0\}$

2 Attractors of *linear* PDEs

2.1 PDEs on finite regions: $x \in [0, l]$

2.1.1 Hamilton systs: Wave and Klein-Gordon

Wave equation: $\ddot{\psi}(x,t) = \psi''(x,t), \qquad x \in [0,l]$

Klein-Gordon: $\ddot{\psi}(x,t) = \psi''(x,t) - m^2\psi(x,t), \qquad x \in [0,l]$

Boundary conditions: $\psi(0,t) = \psi(l,t) = 0$

Hamiltonian form: $\dot{\psi}(x,t) = H_\pi, \qquad \dot{\pi}(x,t) = -H_\psi$

Hamilton functional (real solutions):

$$H(\psi,\pi) = \frac{1}{2}\int_0^l [\pi^2(x) + |\psi'(x)|^2 + m^2\psi^2(x)]dx$$

Phase Space $\mathscr{E} := \{(\psi,\pi) \in H^1(0,l) \oplus L^2(0,l)\}$.

Lemma *Energy conservation*: $H(x(t),v(t)) = \text{const}, \quad t \in \mathbb{R}$

Proof $\dot{H} = \langle H_\psi, \dot{\psi}\rangle + \langle H_\pi, \dot{\pi}\rangle = \langle H_\psi, H_\pi\rangle + \langle H_\pi, -H_\psi\rangle = 0 \quad \square$

Cauchy problem: $\psi(x,0) = \psi_0(x), \ \pi(x,0) = \pi_0(x), \quad x \in [0,l]$

$$\frac{d}{dx}X_k(x) = -\omega_k^2 X_k(x), \ x \in [0,l]; \quad X_k(0) = X_k(l) = 0$$

$X_k = \sin\frac{k\pi}{l}x\,, \quad \omega_k = \frac{k\pi}{l} \qquad k = 1,2,\ldots \qquad \|X_k\|^2_{L^2(0,l)} = l/2$

$\psi(t) = \sum \hat{\psi}(k,t)X_k, \qquad \psi_0 = \sum \hat{\psi}_0(k)X_k, \quad \pi_0 = \sum \hat{\psi}_0(k)X_k$

$\ddot{\hat{\psi}}(k,t) = -\omega_k^2\hat{\psi}(k,t), \quad \hat{\psi}(k,0) = \hat{\psi}_0(k),\ \dot{\hat{\psi}}(k,0) = \hat{\pi}_0(k)$

Solution $\hat{\psi}(k,t) = \hat{\psi}_0(k)\cos\omega_k t + \hat{\pi}_0(k)\dfrac{\sin\omega_k t}{\omega_k}$

Attractor: $\mathscr{A} = \mathscr{E}$ (*exercise*)

2.1.2 Dissipative system: Diffusion equation

$\dot{\psi}(x,t) = \psi''(x,t)$, $x \in [0,l]$ + Dirichlet b.c.

Cauchy problem: $\psi(x,0) = \psi_0(x), \; x \in [0,l]$

Phase Space $\mathscr{E} := L^2(0,l), \quad \psi(t) = \sum \hat{\psi}(k,t) X_k$

$\dot{\hat{\psi}}(k,t) = -\omega_k^2 \hat{\psi}(k,t), \quad \hat{\psi}(k,0) = \hat{\psi}_0(k)$

$\hat{\psi}(k,t) = \hat{\psi}_0(k) e^{-\omega_k^2 t} \to 0 \qquad t \to \infty$

Exercise: Prove

$\|\psi(t)\|_{L^2(0,l)} \leq \|\psi_0\|_{L^2(0,l)} e^{-\omega_1 t}, \quad t \to \infty$

Corollary Attractor $\mathscr{A} = \{0\}$

2.2 PDEs in infinite space: $x \in \mathbb{R}$

Fourier transform: $\tilde{\psi}(\xi) = F\psi := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} \psi(x) dx$

Parseval-Plancherel Theorem: $\|\tilde{\psi}\|^2_{L^2(\mathbb{R})} = \|\psi\|^2_{L^2(\mathbb{R})}$

2.2.1 Diffusion equation

Cauchy problem: $\psi(x,0) = \psi_0(x), \; x \in \mathbb{R}$

Phase Space $\mathscr{E} := L^2(\mathbb{R})$

Solution $\tilde{\psi}(\xi,t) = \tilde{\psi}_0(\xi) e^{-|\xi|^2 t}$

Exrercise: Prove $\|\psi(t)\|_{L^2(\mathbb{R})} \to 0, \quad t \to \infty$

Corollary The attractor $\mathscr{A} = \{0\}$

2.2.2 Wave and Klein-Gordon equations

$\ddot{\psi}(x,t) = \psi''(x,t) - m^2\psi(x,t), \quad x \in \mathbb{R}; \qquad m \geq 0$

Cauchy problem: $\quad \psi(x,0) = \psi_0(x), \;\; \pi(x,0) = \pi_0(x), \quad x \in \mathbb{R}$

Hamilton functional (real solutions):

$$H(\psi,\pi) = \frac{1}{2}\int_{\mathbb{R}} [\pi^2(x) + |\psi'(x)|^2 + m^2\psi^2(x)]dx$$

Phase Space $\;\mathscr{E} := \{(\psi,\pi) \in H^1(\mathbb{R}) \oplus L^2(\mathbb{R})\} \quad$ if $m > 0$.

Solution $\quad \tilde{\psi}(\xi,t) = \tilde{\psi}_0(\xi)\cos\omega(\xi)t + \tilde{\pi}_0(\xi)\dfrac{\sin\omega(\xi)t}{\omega(\xi)}$

the *dispersion relation* $\quad \omega(\xi) := \sqrt{\xi^2 + m^2}$

Exercise: $\forall\,(\psi_0,\pi_0) \in \mathscr{E} \;\; \exists!$ solution $(\psi(t),\pi(t)) \in C(\mathbb{R},\mathscr{E})$

Attractor: $\mathscr{A} = \mathscr{E}$ (*exercise*)

2.3 Local energy decay in infinite space: $x \in \mathbb{R}$

Theorem 1 Let $(\psi_0, \pi_0) \in \mathscr{E}$. Then for any $R > 0$

$$\int_{|x|<R} [\dot\psi^2(x,t) + |\psi'(x,t)|^2 + m^2\psi^2(x,t)]dx \to 0,\ t \to \pm\infty$$

Proof for the case $m = 0$: i) $\psi(x,t) = f(x-t) + g(x+t)$,

$$H(\psi,\dot\psi) = \frac{1}{2}\int_{\mathbb{R}} [|-f'(x-t)+g'(x+t)|^2 + |f'(x-t)+g'(x+t)|^2]dx$$

$$= \int_{\mathbb{R}} [|f'(x-t)|^2 + |g'(x+t)|^2]dx = \int_{\mathbb{R}} [|f'(y)|^2 + |g'(y)|^2]dy < \infty$$

ii) Hence, as $t \to \pm\infty$,

$$\int_{-R}^{R} [\dot\psi^2(x,t) + |\psi'(x,t)|^2]dx = \int_{-R-t}^{R-t} [|f'(y)|^2 dy + \int_{-R+t}^{R+t} |g'(y)|^2]dy \to 0 \quad \square$$

$m > 0$, $\pi_0 = 0$: $\psi(x,t) = C \int e^{-i\xi x} \tilde{\psi}_0(\xi) \cos\omega(\xi)t \, d\xi = \psi_+ + \psi$

$$\psi_+(x,t) = C \int e^{-i\xi x} \tilde{\psi}_0(\xi) e^{i\omega(\xi)t} d\xi = C \int \frac{e^{-i\xi x}\tilde{\psi}_0(\xi)}{\omega'(\xi)t} de^{i\omega(\xi)t}$$

Assume $\tilde{\psi}_0 \in C_0^\infty(\mathbb{R})$ and $\tilde{\psi}_0(\xi) \equiv 0$ for $|\xi| < \varepsilon$. Then

$$|\psi_+(x,t)| \le C_1 \frac{|x|+1}{|t|} \Longrightarrow |\psi_+(x,t)|^2 \le 2C_1^2 \frac{|x|^2+1}{|t|^2}$$

since $|\omega'(\xi)| \ge \varkappa(\varepsilon) > 0$ for $|\xi| \ge \varepsilon > 0$.

Hence, $\forall R > 0$: $\displaystyle\int_{|x|<R} |\psi_+(x,t)|^2 dx \le C_2 \frac{R^3+1}{|t|^2} \to 0, \ t \to \pm\infty$

General case $\psi_0 \in H^1(R)$: $\forall \varepsilon > 0$ $\psi_0(x) = \varphi(x) + h(x)$,

$\tilde{\varphi} \in C_0^\infty(\mathbb{R})$, $\tilde{\varphi}(\xi) \equiv 0$ for $|\xi| < \varepsilon$, $\quad \|h\|_{H^1(\mathbb{R})} \le \varepsilon$

$$\int_{|x|<R} |\psi_\varphi(x,t)|^2 dx \le C_3 \frac{R^3+1}{|t|^2}, \quad \|\psi_h(t)\|_{H^1(\mathbb{R})} \le C_4 \varepsilon \quad \square$$

2.4 Attractors in local seminorms

2.4.1 Klein-Gordon equation ($m > 0$)

Definition *The set of stationary states:*

$$\mathscr{S} = \{(\psi, 0) : \psi''(x) - m^2 \psi(x) = 0,\ x \in \mathbb{R};\ H(\psi, 0) < \infty\}$$

Exercise: Check i) $\mathscr{S} = \{(0,0)\}$ for $m > 0$

ii) $\mathscr{S} = \{(C,0) : C \in \mathbb{R}\}$ for $m = 0$

Definiton $\mathscr{E}_{\rm loc}$ is $\mathscr{E}$ with topology $H^1_{\rm loc}(\mathbb{R}) \oplus L^2_{\rm loc}(\mathbb{R})$

Corollary $\mathscr{A} = \{0\}$ in phase space $\mathscr{E}_{\rm loc}$:

$$Y(t) := (\psi(t), \pi(t)) \xrightarrow{\mathscr{E}_{\rm loc}} 0.$$

2.4.2 Wave equation ($m = 0$)

Proposition $\mathscr{A} = \mathscr{S}$ in phase space $\mathscr{E}_{\rm loc}$.

Proof: By Theorem 1,

$$\int_{|x|<R} [\pi^2(x,t) + |\psi'(x,t)|^2] dx \to 0, \quad t \to \pm\infty$$

Define $S(t) = (\psi(0,t), 0) \in \mathscr{S}$. Then

$$\|(\psi(t), \pi(t)) - S(t)\|^2_{H^1(-R,R)\oplus L^2(-R,R)} = \|\psi(t) - \psi(0,t)\|^2_{H^1(-R,R)}$$

$$= \int_{|x|<R} [|\psi'(x,t)|^2 + |\psi(x,t) - \psi(0,t)|^2 + |\pi(x,t)|^2] dx,$$

$$|\psi(x,t) - \psi(0,t)|^2 = \Big|\int_0^x \psi'(y,t) dy\Big|^2$$

$$\le |x| \int_0^x \big|\psi'(y,t)\big|^2 dy \le |R| \int_{-R}^R |\psi'(y,t)|^2 dy. \quad \square$$

3 Linear Schrödinger Eqn

Linear Schrödinger Eqn

$i\dot{\psi}(x,t) = \Delta\psi(x,t) + V(x)\psi(x,t),\ x \in \mathbb{R}^n;\quad \psi(\cdot,0) \in L^2(\mathbb{R}^n)$

If the potential $V(x)$ is sufficiently smooth

and decays sufficiently rapidly as $|x| \to \infty$,

$\psi(x,t) = \sum_1^N \psi_k(x) e^{-i\omega_k t} + w(x,t)$

$$w(\cdot,t) \xrightarrow{L^2{}_{\mathrm{loc}}(\mathbb{R}^n)} 0, \qquad t \to \pm\infty$$

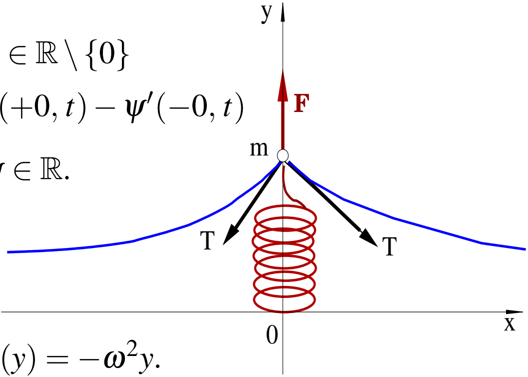
Corollary Attractor $\mathscr{A} = (\psi_1, ..., \psi_N) \equiv \mathbb{C}^N$ in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$

4 Attractors and scattering in nonlinear Lamb system

A. Komech, A. Merzon: Scattering in the nonlinear Lamb System, *Physics Letters A*, **373** (2009), 1005-1010

$$\begin{cases} \ddot{\psi}(x,t) = \psi''(x,t), \quad x \in \mathbb{R} \setminus \{0\} \\ m\ddot{y}(t) = F(y(t)) + \psi'(+0,t) - \psi'(-0,t) \end{cases}$$

$y(t) \equiv \psi(0,t), m > 0,\ \psi \in \mathbb{R}.$



H. Lamb (1900): for $F(y) = -\omega^2 y$.

$Z := \{C \in \mathbb{R} : F(C) = 0\}, \quad Y(t) := (\psi(\cdot,t), \dot\psi(\cdot,t), \dot y(t))$

Set of Stationary States $\mathscr{S} = \{S = (C,0,0) : C \in Z\}$

$\psi|_{t=0} = u_0(x),\ \dot\psi|_{t=0} = \pi_0(x),\ \dot y|_{t=0} = p_0$

$\dot Y(t) = \mathscr{F}(Y(t))$ for $t \in \mathbb{R}$

$Y(0) = Y_0 = (\psi_0, \pi_0, p_0)$

Hilbert phase space

$\mathscr{E} := \{Y = (\psi, \pi, p) : \psi', \pi \in L^2(\mathbb{R}), p \in \mathbb{R})\},$

$\|Y\|_{\mathscr{E}} := \|\psi\|_{L^2(\mathbb{R})} + |\psi(0)| + \|\pi\|_{L^2(\mathbb{R})} + |p|.$

$\mathscr{E}_{\rm loc}$ is the space $\mathscr{E}$ with **local energy seminorms**

$\|(\psi, \pi, p)\|_{\mathscr{E},R} \equiv \|\psi'\|_R + |\psi(0)| + \|\pi\|_R + |p|, \ \ R > 0.$

Assumptions:

$F(\psi) = -V'(\psi)$

$V(\psi) \to +\infty, \quad |\psi| \to \infty$

$$\mathscr{H}(\psi,\pi,p) = \frac{1}{2}\int [|\pi(x)|^2 + |\psi'(x)|^2]dx + m\frac{|p|^2}{2} + V(\psi(0))$$

Proposition *For any* $Y_0 \in \mathscr{E}$ $\exists$ *a unique solution* $Y(t) \in C(\mathbb{R}, \mathscr{E})$,

and the energy is conserved: $\quad H(Y(t)) = \mathrm{const},\ t \in \mathbb{R}$

Theorem 1 *Let a solution* $Y(t) \in C(\mathbb{R}, \mathscr{E})$. *Then*

i) $Y(t) \xrightarrow{\mathscr{E}_{\rm loc}} \mathscr{S}, \qquad t \to \pm\infty$

ii) Let Z *be discrete in* $\mathbb{R}$. *Then*

$Y(t) \xrightarrow{\mathscr{E}_{\rm loc}} S_\pm \in \mathscr{S},\ t \to \pm\infty$

Theorem 2 *Let a solution* $Y(t) \in C(\mathbb{R}, \mathscr{E})$ *and*

$$\exists\ \psi_0^\pm := \lim_{x \to \pm\infty} \psi_0(x), \quad I_0 := \int_{\mathbb{R}} |\pi_0(y)| dy < \infty$$

Then there exist $S_\pm = (C_\pm, 0, 0) \in \mathscr{S}$ *and* $\Psi_\pm \in \mathscr{E}$,

$$Y(t) = S_\pm + W(t)\Psi_\pm + r_\pm(t), \quad \|r_\pm(t)\|_{\mathscr{E}} \to 0,\ t \to \pm\infty$$

$W(t)$ *is the dynamical group of* **free wave equation**

Proof: For $\pm x > 0$ the solution admits the d'Alembert representation

$$\psi(x,t) = f_\pm(x-t) + g_\pm(x+t)$$

$$\dot\psi(x,t) = -f'_\pm(x-t) + g'_\pm(x+t), \quad \psi'(x,t) = f'_\pm(x-t) + g'_\pm(x+t)$$

$f'_\pm, g'_\pm \in L^2(\mathbb{R}, \mathbb{R}^d)$ since $(\psi_0, \pi_0) \in \mathscr{E}$.

For $\pm z > 0$, the d'Alembert formulas

$$f_\pm(z) := \frac{\psi_0(z)}{2} - \frac{1}{2}\int_0^z \pi_0(y)\,dy, \quad g_\pm(z) := \frac{\psi_0(z)}{2} + \frac{1}{2}\int_0^z \pi_0(y)\,dy$$

give the solution for $|x| > |t|$.

For $|x| < |t|$ we need "reflected waves": $f_+(z), z < 0; g_-(z), z > 0$.

“gluing condition” at $x=0$:

$$y(t) := \psi(0,t) = f_+(-t) + g_+(t) = f_-(-t) + g_-(t)$$

$$f_+(-t) := y(t) - g_+(t),\ g_-(t) := y(t) - f_-(-t),\ \ t>0$$

$$\psi(x,t) = \left\{ \begin{array}{ll} y(t-x) + g_+(x+t) - g_+(t-x), & 0<x<t \\ y(t+x) + f_-(x-t) - f_-(-x-t), & -t<x<0 \end{array} \right| \ t>0$$

$$\ddot{y}(t) = F(y(t)) - 2\dot{y}(t) + 2\dot{w}_{\rm in}(t),\ t>0; \quad y(0)=\psi_0(0),\ \dot{y}(0)=p_0$$

$$w_{\rm in}(t) = g_+(t) + f_-(-t), \quad \dot{w}_{\rm in} \in L^2(\mathbb{R}).$$

A priori bound

$$\sup_{t>0}|y(t)| + \sup_{t>0}|\dot{y}(t)| + \int_0^\infty |\dot{y}(t)|^2 dt \le B(Y_0) < \infty$$

Corollary i) $y(t) \to Z \subset \mathbb{R}$ and $\dot{y}(t) \to 0$ as $t \to +\infty$;

ii) moreover, $y(t) \to z_+ \in Z$ as $t \to +\infty$ if Z is discrete ... □