On global attractors of Hamilton nonlinear PDEs

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Abstract

Theory of attractors for Hamilton nonlinear PDEs was developed since 1995 together with P. Joly and H. Spohn and since 2006 together with A. Comech, V. Imaikin, E. Kopylova and B. Vainberg.

Survey:

A. Komech, Attractors of nonlinear Hamilton PDEs, *Discrete and Continuous Dynamical Systems A* **36** (2016), no. 11, 6201-6256.

The theory was inspired by fundamental phenomena of classical and quantum physics: radiation damping in classical electrodynamics, Bohr's transitions to quantum stationary states, L. de Broglie's wave-particle duality, and others.

Survey:

A. Komech, Quantum Mechanics: Genesis and Achievements, Springer, Dordrecht, 2013

Plan

I. Physical motivations

II. Attraction to stationary states

III. Attraction to solitons

IV. Adiabatic effective dynamics of solitons

V. Attraction to stationary orbits

VI. Open problems

VII. General conjecture for G-invariant equations

VIII. Comparison with attractors for dissipative systems

1913 N. Bohr: Quantum Transitions

(QT) $|E_-\rangle \mapsto |E_+\rangle$ Quantum Stationary Orbits

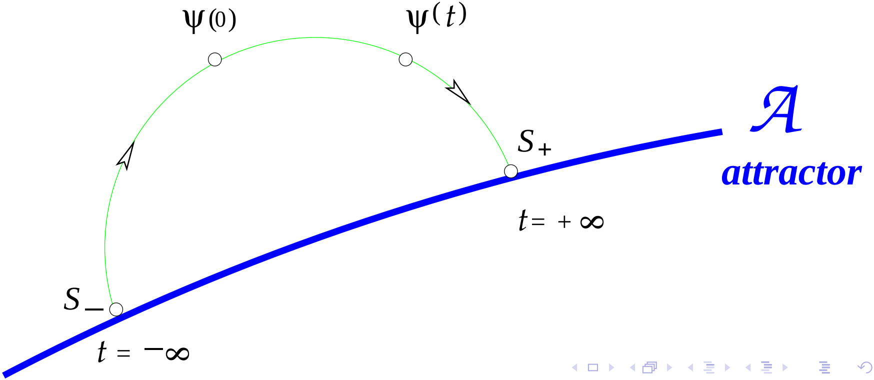
I. Physical motivations

1913 N. Bohr: Quantum Transitions

(QT) $|E_-\rangle \mapsto |E_+\rangle$ Quantum Stationary Orbits

Dynamical interpretation: Long-time attraction

(B) $\psi(t) \sim S_\pm, \qquad t \to \pm\infty$



Example

$$\ddot{\psi}(x,t)=\psi''(x,t),\quad \psi(x,0)=\psi_0(x),\quad \pi(x,0)=\pi_0(x),\qquad x\in\mathbb{R}$$

Let $\exists u_0^\pm := \lim\limits_{x\to\pm\infty} u_0(x)$ and $I_0 := \int_{\mathbb{R}} |\pi_0(y)|dy < \infty$

Then

$$Y(t)\xrightarrow{\mathcal{E}_{\rm loc}} S_\pm=(C_\pm,0),\quad t\to\pm\infty$$

Proof:

$$\psi(x,t)=\frac{\psi_0(x-t)+\psi_0(x+t)}{2}+\frac{1}{2}\int\limits_{x-t}^{x+t}\pi_0(y)dy \qquad \square$$

Exercise: Prove that such transitions exist $\forall (C_\pm,0)$.

1926 E. Schrödinger: Quantum Mechanics

S1 Equation: $i\dot\psi(x,t) = H\psi(x,t), \qquad H = \Delta + V(x)$

S2 Quantum Stationary Orbits: $|E\rangle \Longleftrightarrow$ *solutions* $\psi(x)e^{-i\omega t}, \quad \omega = E/\hbar$

Dynamical interpretation: Attraction to Quantum Stationary Orbits

$$(S) \qquad \psi(x,t) \sim \psi_\pm(x)e^{-i\omega_\pm t}, \qquad t \to \pm\infty$$

Such attraction is *impossible* for **linear** Shrödinger equation **S1**

since the sum $C_1\psi_1(x)e^{-i\omega_1 t} + C_2\psi_2(x)e^{-i\omega_2 t}$ is also a solution !

We suggest (S) hold for **nonlinear** coupled Maxwell-Schrödinger Eqns

$$\begin{cases} [i\partial_t - V(\mathbf{x},t) - V_{\rm ext}(\mathbf{x},t)]\psi(\mathbf{x},t) = [-i\nabla - \mathbf{A}(\mathbf{x},t) - \mathbf{A}_{\rm ext}(\mathbf{x})]^2\psi(\mathbf{x},t) \\ \Box V(\mathbf{x},t) = \rho(\mathbf{x},t) = |\psi(\mathbf{x},t)|^2, \qquad \Box := \partial_t^2 - \Delta \\ \Box \mathbf{A}(\mathbf{x},t) = \mathbf{j}(\mathbf{x},t) = \mathrm{Im}\,\{\overline{\psi(\mathbf{x},t)}[-i\nabla - \mathbf{A}(\mathbf{x},t) - \mathbf{A}_{\rm ext}(\mathbf{x})]\psi(\mathbf{x},t)\} \end{cases}$$

1924 L. de Broglie: Wave-Particle duality

beam of particles with momentum P and energy E

$$\Longleftrightarrow \qquad \text{wave } \psi(x,t) = Ce^{i(px-\omega t)}, \qquad p = P/\hbar, \ \ \omega = E/\hbar$$

Dynamical interpretation: Long-time attraction to solitons

$$(WP) \qquad \psi(x,t) \sim \psi_\pm(px-\omega t), \qquad t \to \pm\infty$$

Such attraction is proved for *Korteweg de Vries Eqn* and other *integrable Eqns* by Ablowitz, Segur, Eckhaus, Its, ... by *Method of Inverse Scattering*

II. Attraction to stationary states

$$\ddot{\psi}(x,t)=\psi''(x,t)+f(x,\psi(x,t)),\quad x\in\mathbb{R}$$

$$f(x,\psi)=-\nabla_\psi U(x,\psi),\qquad x\in\mathbb{R},\ \psi\in\mathbb{R}^N$$

i) $U(x,\psi)\geq -C$, ii) $\min\limits_{x\in[a,b]} U(x,\psi)\to\infty$ as $|\psi|\to\infty$ (U)

iii) $U(x,\psi)=0$ for $|x|\geq c$.

The Hamilton functional

$$H=\frac{1}{2}\int[|\dot{\psi}(x,t)|^2+|\psi'(x)|^2]dx+\int U(x,\psi(x))dx$$

The Hamilton form

$$\dot{\psi}(x,t)=\pi(x,t),\quad \dot{\pi}(x,t)=\psi''(x,t)+f(x,\psi(x,t)),\quad x\in\mathbb{R}$$

The set of stationary states $\mathcal{S} := \{\psi : \psi''(x) + f(x, \psi(x)) \equiv 0\}$

Theorem 1. (K. 1999) i) Let (U) hold. Then

$$\mathrm{dist}_{H^1_{\mathrm{loc}}(\mathbb{R})}((\psi(\cdot, t), \mathcal{S}) \to 0, \qquad t \to \pm\infty. \qquad (A1)$$

ii) Let $f(x, \psi)$ be real analytic in $\psi \in \mathbb{R}^N$. Then the set S is discrete and

$$\psi(\cdot, t) \xrightarrow{H^1_{\mathrm{loc}}(\mathbb{R})} S_\pm \in \mathcal{S}, \qquad t \to \pm\infty. \qquad \text{(A2)}$$

Proof: Energy radiation to infinity.

Couterexample. (A2) can fail if the set $\mathcal{S}$ **is not discrete:** Let

$$f(x, \psi) \equiv 0 \quad \text{for} \quad x \in \mathbb{R}, \quad \psi \in [-1, 1].$$

Then (A2) fails for the solution $\psi(x, t) = \sin[\log(|x - t| + 2)]$

3D wave-particle system

$$\begin{cases} \ddot\psi(x,t) = \Delta\psi(x,t) - \rho(x-q(t)), \quad x\in\mathbb{R}^3 \\ \dot q = p/\sqrt{1+p^2}, \qquad \dot p = -\nabla V(q) - \int \nabla\psi(x,t)\rho(x-q)\,dx \end{cases}$$

$\rho \in C_0^\infty(\mathbb{R}^3)$, $\quad q\in\mathbb{R}^3$. $\qquad V(q)\to\infty$ as $|q|\to\infty$

Wiener condition (FGR): $\quad \hat\rho(k) := \int e^{ikx}\rho(x)dx \neq 0, \quad k\in\mathbb{R}^3 \quad (W)$

(H) $\quad H = \frac{1}{2}\int [|\pi(x)|^2 + |\nabla\psi(x)|^2]\,dx + \int \psi(x)\rho(x-q)\,dx + \sqrt{1+p^2} + V(q)$

Phase space $\mathcal{E} := \dot H^1(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)\oplus\mathbb{R}^3\oplus\mathbb{R}^3$

Set of stationary states

$$\mathcal{S} := \{(\psi_q, 0, q, 0) \in \mathcal{E} : \Delta\psi_q(x) - \rho(x) \equiv 0, \nabla V(q) = 0\}.$$

Coulomb potential

$$\psi_q(x) = -\frac{1}{4\pi}\int \frac{\rho(y)dy}{|x-y|}$$

Denote $Y(t) = (\psi(t), \pi(t), q(t), p(t))$

Theorem 2. (K, Spohn, Kunze 1997)

i) $\operatorname{dist}(Y(t), \mathcal{S}) \to 0, \;\; t \to \pm\infty.$

ii) Let the set $\{q \in \mathbb{R}^3 : \nabla V(q) = 0\}$ be discrete in $\mathbb{R}^3$. Then the set $\mathcal{S}$ is discrete in $\mathcal{E}$ and

$$\psi(\cdot, t) \to S_\pm \in \mathcal{S}, \qquad t \to \pm\infty.$$

Proof: Energy radiation and Wiener Tauberian Theorem.

3D Maxwell-Lorentz Eqs. Abraham *extended electron*:

$$\begin{cases} \dot E = \mathrm{rot}\ B - \dot q \rho(x-q), \ \ \dot B = -\mathrm{rot}\ E, \ \ \mathrm{div}\ E = \rho(x-q), \ \ \mathrm{div}\ B = 0 \\ \dot q = \dfrac{p}{\sqrt{1+p^2}}, \ \ \dot p = \displaystyle\int [E + E^{\mathrm{ext}} + \dot q(t) \wedge (B + B^{\mathrm{ext}})]\rho(x - q(t))\, dx \end{cases}$$

Scalar potential $E^{\mathrm{ext}}(x) = -\nabla\phi^{\mathrm{ext}}(x)$, $V(q) := \displaystyle\int \phi^{\mathrm{ext}}(x)\rho(x-q)\, dx$

Hamiltonian $H = \dfrac{1}{2}\displaystyle\int [E^2(x) + B^2(x)]\, dx + V(q) + \sqrt{1+p^2}$

Theorem 3. (K, H. Spohn 2000)

i) Let $V(q) \to \infty$ as $|q| \to \infty$, and (W) hold. Then

$$\mathrm{dist}\,(Y(t), \mathcal{S}) \to 0, \ \ t \to \pm\infty.$$

ii) Let the set $\{q \in \mathbb{R}^3 : \nabla V(q) = 0\}$ be discrete in $\mathbb{R}^3$. Then the set $\mathcal{S}$ is discrete in $\mathcal{E} := L^2 \oplus L^2 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ and $\psi(\cdot, t) \to S_\pm \in \mathcal{S}, \ \ t \to \pm\infty$.

III. Attraction to solitons

Translation-invariant wave-particle system: $V(q) \equiv 0$

$$\begin{cases} \ddot{\psi}(x,t) = \Delta\psi(x,t) - \rho(x - q(t)), \quad x \in \mathbb{R}^3 \\ \dot{q} = p/\sqrt{1+p^2}, \;\; \dot{p} = -\int \nabla\psi(x,t)\rho(x-q)\,dx \end{cases}$$

Momentum conservation: $P(t) := -\int \dot{\psi}(x,t)\nabla\psi(x,t)dx + p(t) = \mathrm{const}$

Solitons: $\psi(x,t) = \psi_v(x - vt - a), \quad q(t) = vt + a.$

Solitary manifold $\mathcal{S} := \{\psi_v(x-a) : |v| < 1, a \in \mathbb{R}^3\}$

Theorem 4. (K, H. Spohn 1998) Let (W) hold. Then

$\psi(x,t) \sim \psi_{v_\pm}(x - q(t)), \quad q(t) \sim v_\pm t + \mathcal{O}(\log t), \qquad t \to \pm\infty$

$\dot{q}(t) \to v_\pm, \qquad \ddot{q}(t) \to 0$ **Radiation Damping**

Extension to translation-invariant ML Eqs: K, V. Imaikin 2004

IV. Adiabatic effective dynamics of solitons

Equation for solitons $(v\nabla)^2\psi_v(y) = \Delta\psi_v(y) - \rho(y), \quad y \in \mathbb{R}^3$

Solution $\hat{\psi}_v(k) = -\dfrac{\hat{\rho}(k)}{k^2 - (vk)^2}, \qquad \psi_v \in \dot{H}^1(\mathbb{R}^3),\ |v| < 1$

Motion of solitons in slowly varying potential $V(\varepsilon q)$, $|\varepsilon| \ll 1$:

$$\psi(x,t) \approx \psi_{v(t)}(x - q(t))$$

Effective Hamiltonian, see (H): $H^{\text{eff}}(P,Q) := H(\psi_v, \pi_v, Q, p_v)$,

where $P = -\int \pi_v \nabla\psi_v + p_v, \ \ \pi_v := -v\nabla\psi_v, \quad p_v := v/\sqrt{1+v^2}$

Effective Dynamics: $\dot{Q} = H^{\text{eff}}_P, \ \ \dot{P} = -H^{\text{eff}}_Q$

Theorem 5. (K, M. Kunze, H. Spohn 1999). Let

$\psi(x,0) = \psi_{v_0}(x),\ \dot{\psi}(x,0) = \pi_{v_0}(x),\ \ x \in \mathbb{R}^3;\ Q(0) = q(0),\ P(0) = P_{v_0}$

Then $|q(t) - Q(t)| \le C_0, \quad |t| \le C\varepsilon^{-1}$

V. Attraction to 'stationary orbits' $\psi(x)e^{i\omega t}$

1D KG Eqn coupled to nonlinear oscillator

$$\ddot\psi(x,t)=\psi''(x,t)-m^2\psi(x,t)+\delta(x)f(\psi(0,t)),\quad x\in\mathbb{R}\qquad (KG)$$

C1 $f(\psi)=-\nabla U(\psi),\qquad \inf U(\psi)>-\infty$

Hamiltonian $(\pi(x,t):=\dot\psi(x,t))$

$$H=\frac12\int[|\pi(x)|^2+|\psi'(x)|^2+m^2|\psi(x)|^2]dx+U(\psi(0))$$

C2 **$U(1)$-invariance:** $U(\psi)=u(|\psi|)$

Solitary waves $\psi(x,t)=\psi_\omega(x)e^{i\omega t}$, $\omega\in\Omega\subset(-m,m)$

Nonlinear eigenvalue problem

$$-\omega^2\psi_\omega(x)=\psi_\omega''(x)-m^2\psi_\omega(x)+\delta(x)f(\psi_\omega(0))$$

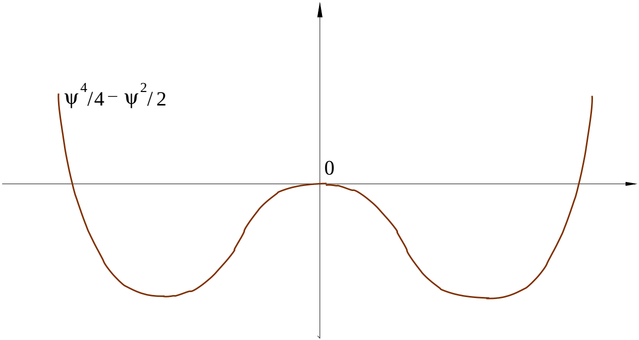
$$\psi_\omega(x)=Ce^{-\varkappa|x|},\qquad 2\varkappa C=f(C),\qquad \varkappa:=\sqrt{m^2-\omega^2}>0$$

Solitary manifold $\mathcal{S}=\{e^{i\theta}\psi_\omega(x):\omega\in\Omega,\ \theta\in[0,2\pi]\}$.

C3 Equation (KG) is Strictly Nonlinear:

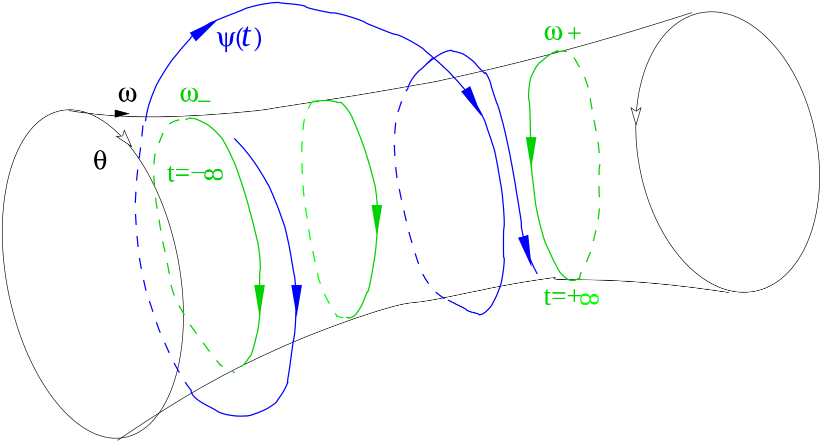
$$U(\psi)=u(|\psi|)=\Sigma_0^N u_j|\psi|^{2j},\quad u_N>0,\quad N\geq 2 \qquad \text{(SNL)}$$

Example: *Ginzburg-Landau potential* $U(\psi)=\psi^4/4-\psi^2/2$, $f(\psi)=-|\psi|^2\psi+\psi$



Theorem 6. (K. 2003) *Let* **C1 – C3** *hold. Then*

$$\psi(\cdot, t) \xrightarrow{H^1_{\rm loc}(\mathbb{R})} \mathcal{S}, \quad t \to \pm\infty \qquad \forall \textit{ solution } \psi \in C(\mathbb{R}, H^1(\mathbb{R})).$$



Methods

The proof relies on a novel strategy:

(1) Fourier-Laplace transform in time.

(2) Nonlinear analogue of Kato's theorem

on absence of embedded eigenvalues.

(3) Reduction of spectrum of omega-limit

trajectories to the spectral gap.

(4) Reduction of the spectrum to a single point.

ω-limit trajectories and Fourier transform in time

ω-limit trajectories: $\psi(\cdot, s_k + t) \xrightarrow{\mathcal{E}_{\rm loc}} \beta(\cdot, t), \;\; s_k \to \infty, \qquad t \in \mathbb{R}$

Example: $\psi(x,t) \sim \psi_*(x)e^{i\omega_* t} \Longrightarrow \beta(x,t) = e^{i\theta}\psi_*(x)e^{i\omega_* t}$

Theorem 6 $\iff$ Each omega-limit trajectory $\beta(x,t) \equiv \psi_*(x)e^{i\omega_* t}$

Equivalenly: Fourier transform in time $\quad \mathrm{supp}\ \tilde\beta(x,\cdot) = \{\omega_*\}$

Binary Radiation Mechanism:

I. *linear dispersion* and **II.** *nonlinear inflation of spectrum*

I. Linear dispersion

Klein-Gordon eqn with *harmonic source*

$$\ddot{\psi}(x,t) = \psi''(x,t) - m^2\psi + C\delta(x)e^{i\omega_0 t}, \qquad |\omega_0| > m$$

Limiting Amplitude Principle: $\psi(x,t) \sim a(x)e^{i\omega_0 t}, \quad t \to \infty$

$$-\omega_0^2 a(x) = a''(x) - m^2 a(x) + C\delta(x) \implies \hat{a}(k) = \frac{C}{k^2 - (\omega_0^2 - m^2)}$$

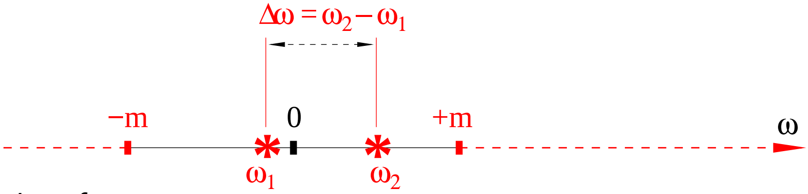
$$|\omega_0| > m \implies \hat{a}(k) \notin L^2(\mathbb{R}) \implies \|\phi(t)\|_{H^1} \to \infty, \quad t \to \infty$$

Corollary $\operatorname{supp} \tilde{\beta}(x,\cdot) \subset [-m,m]$ for each ω-limit trajectory

II. Nonlinear inflation of spectrum

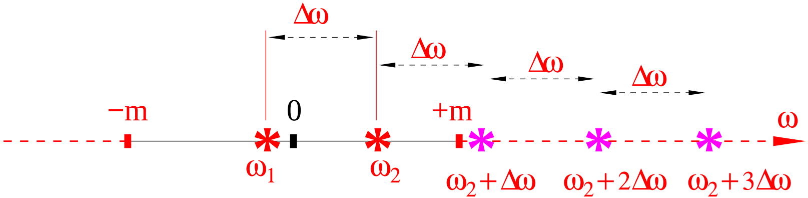
Example: $U(|\psi|^2) = |\psi|^4$: then $f(\psi) = -\nabla_\psi U(|\psi|^2) = -4|\psi|^2\psi$

Substitute $\psi(0,t) = e^{i\omega_1 t} + e^{i\omega_2 t}$ with two point spectrum



Inflation of spectrum:

$f(\psi(0,t)) \sim \psi\overline{\psi}\psi = e^{i\omega_2 t} e^{-i\omega_1 t} e^{i\omega_2 t} + \ldots = e^{i(\omega_2 + \Delta\omega)t} + \ldots$



Separation of dispersion component

(KG) $\ddot{\psi}(x,t) = \Delta\psi - m^2\psi + \delta(x) f_0(t)$, where $f_0(t) := f(\psi(0,t))$

A priori est: Energy conservation $\Longrightarrow$ $f_0 \in C_b(\mathbb{R})$, $\psi \in C_b(\mathbb{R}, H^1(\mathbb{R}))$

Splitting $\psi = w + \varphi$**:**

$$\ddot{w} = w'' - m^2 w, \qquad w(x,0) = \psi(x,0), \dot{w}(x,0) = \dot{\psi}(x,0), \quad w \in C_b(\mathbb{R}, H^1(\mathbb{R}))$$

$$\ddot{\varphi} = \varphi'' - m^2\varphi + f_0, \quad \varphi(x,0) = 0, \qquad \dot{\varphi}(x,0) = 0, \qquad \varphi \in C_b(\mathbb{R}, H^1(\mathbb{R}))$$

$$w(\cdot,t) \xrightarrow{H^1_{\rm loc}(\mathbb{R})} 0, \qquad t \to \infty$$

It remains to prove: $\varphi(\cdot,t) \xrightarrow{H^1_{\rm loc}(\mathbb{R})} \mathcal{S}, \qquad t \to +\infty$

Fourier-Laplace transform

Our plan: **I.** $\operatorname{supp}\tilde\beta(x,\cdot)\subset[-m,m]$ **II.** $\operatorname{supp}\tilde\beta(x,\cdot)=\{\omega_*\}$

Fourier-Laplace: $\tilde\varphi^+(x,\omega):=\int_0^\infty e^{i\omega t}\varphi(x,t)dt$, $\quad\mathrm{Im}\,\omega>0$

$$-\omega^2\tilde\varphi^+=\partial_x^2\tilde\varphi^+-m^2\tilde\varphi^++\tilde f_0(\omega),\quad \omega\in\mathbb{C}^+$$

$$\tilde\varphi^+(x,\omega)=-\tilde f_0^+(\omega)\frac{e^{ik(\omega)|x|}}{2ik(\omega)}=\tilde g(\omega)e^{ik(\omega)|x|};\qquad k(\omega):=\sqrt{\omega^2-m^2}\in\mathbb{C}^+$$

Remark: $g(t)=\varphi^+(0,t)\in L^\infty(\mathbb{R})$.

Lemma. $\tilde\varphi^+(x,\omega+i0)=\tilde g(\omega+i0)e^{ik(\omega)|x|},\quad \omega\in\mathbb{R},$

the multiplication in the sense of **quasimeasures**

Corollary. $\varphi(x,t)=\dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}}e^{-i\omega t}\tilde g(\omega+i0)e^{ik(\omega)|x|}d\omega,\ \ t>0$

Quasimeasures

$$\check{g}(t) := \mathcal{F}^{-1}_{\omega\to t}[g(\omega)]$$

Definition $\mu(\omega) \in S'(\mathbb{R})$ is a *quasimeasure* if $\check{\mu} \in L^\infty(\mathbb{R})$.

Definition $\mathcal{A}(\mathbb{R})$ is the space of $f(t) \in L^\infty(\mathbb{R})$: $f_\varepsilon(t) \xrightarrow{\mathcal{A}} f(t)$, $\varepsilon \to 0$

1. $\forall T > 0, \quad \|f_\varepsilon(t) - f(t)\|_{L^\infty(-T,T)} \to 0$, $\varepsilon \to 0$
2. $\sup\limits_{|\varepsilon|<1} \|f_\varepsilon(t)\|_{L^\infty(\mathbb{R})} < \infty$

Definition. $\mathcal{QM}(\mathbb{R}) := F^{-1}\mathcal{A}(\mathbb{R})$ is the space of all quasimeasures $\mu(\omega)$

$$\mu_\varepsilon(\omega) \xrightarrow[\varepsilon\to 0]{\mathcal{QM}} \mu(\omega) \quad \Longleftrightarrow \quad \check{\mu}_\varepsilon(t) \xrightarrow[\varepsilon\to 0]{\mathcal{A}} \check{\mu}(t)$$

Multiplicators of Quaimeasures

Let $M(\omega) \in C(\mathbb{R})$ and $M: \ \mu(\omega) \mapsto M(\omega)\mu(\omega)$

Lemma

1. Let $\check{M}(t) \in L^1(\mathbb{R})$. Then $\quad M: \ \mathcal{QM}(\mathbb{R}) \to \mathcal{QM}(\mathbb{R})$
2. Let $\mu_\varepsilon(\omega) \stackrel{\mathcal{QM}}{\longrightarrow} \mu(\omega)$ and $\check{M}_\varepsilon(t) \stackrel{L^1}{\longrightarrow} \check{M}(t)$ as $\varepsilon \to 0$. Then

$$M_\varepsilon(\omega)\mu_\varepsilon(\omega) \stackrel{\mathcal{QM}}{\longrightarrow} M(\omega)\mu(\omega), \qquad \varepsilon \to 0$$

Examples: $x \in \mathbb{R}, \qquad k(\omega) = \sqrt{\omega^2 - m^2}$

i) $M(\omega) = \zeta(\omega)e^{ik(\omega)|x|}$

ii) $M_\varepsilon(\omega) = \zeta(\omega)e^{ik(\omega+i\varepsilon)|x|}$

Nonlinear Kato's theorem

Second splitting: $\zeta(\omega) \in C_0^\infty(\mathbb{R})$, $\zeta(\omega) \equiv 1$ for $|\omega| \le m$

$$\varphi(x,t) = \frac{1}{2\pi}\int (1-\zeta(\omega))e^{-i\omega t}...d\omega + \frac{1}{2\pi}\int \zeta(\omega)e^{-i\omega t}...d\omega$$

$$= \varphi_d(x,t) + \varphi_b(x,t), \;\; t > 0$$

$$\ddot\varphi(x,t) = \varphi''(x,t) + f_0(t), \qquad \varphi \in C_b(\mathbb{R}, H^1(\mathbb{R})), \quad f_0 \in C_b(\mathbb{R}).$$

Theorem A. $$\int_{|\omega|>m} |\tilde g(\omega)|^2 \omega k(\omega) d\omega < \infty$$

Corollary: Local Energy Decay $\quad \varphi_d(\cdot,t) \xrightarrow{H^1_{\rm loc}(\mathbb{R})} 0, \;\; t \to +\infty$

It remains to prove $\quad \varphi_b(\cdot,t) \xrightarrow{H^1_{\rm loc}(\mathbb{R})} \mathcal{S}, \;\; t \to +\infty$

Compactness and Spectral Identity

$$\varphi_b(x,t)=\frac{1}{2\pi}\int e^{-i\omega t}\zeta(\omega)\tilde g(\omega)e^{ik(\omega)|x|}d\omega$$

Lemma. $\sup_{0<|x|\le R}\ \sup_{t\in\mathbb{R}}|\partial_x^j\partial_t^l\varphi_b(x,t)|<\infty,\ \forall R,j,l>0$

Corollary. $\forall s_j\to+\infty$: $\varphi_b(x,s_{j'}+t)\to\beta(x,t)$ for $x,t\in\mathbb{R}$,

$$\tilde\beta(x,\omega)=\lim_{j'\to\infty}\tilde\varphi_b(x,\omega)e^{-i\omega s_{j'}}=\lim_{j'\to\infty}\zeta(\omega)\tilde g(\omega)e^{ik(\omega)|x|}e^{-i\omega s_{j'}}$$

$$=e^{ik(\omega)|x|}\lim_{j'\to\infty}\zeta(\omega)\tilde g(\omega)e^{-i\omega s_{j'}}=:e^{ik(\omega)|x|}\tilde\gamma(\omega),\quad\forall x\in\mathbb{R}$$

in the sense of *quasimeasures* **of** $\omega\in\mathbb{R}$. *Remark:* $\beta(0,t)=\gamma(t)$.

Lemma. $\operatorname{supp}\tilde\gamma\subset[-m,m]$. **Proof:** Thm A + Riemann-Lebesgue Thm. □

Spectral Identity $\operatorname{supp}\tilde\beta(x,\cdot)=\operatorname{supp}\tilde\gamma\subset[-m,m],\ \forall x\in\mathbb{R}$.

Equation for ω-limit trajectories

Lemma. $\beta(x,t)$ is an ω-limit trajectory:

Proof: $\psi(\cdot, t+s_{j''}) = w + \varphi_d + \varphi_b \xrightarrow{H^1_{\rm loc}(\mathbb{R})} \beta, \qquad t \in \mathbb{R}.$ $\square$

Lemma. $\ddot\beta(x,t) = \beta''(x,t) - m^2\beta(x,t) + \delta(x) f(\beta(0,t)), \quad x,t \in \mathbb{R}$

Proof:

$$\ddot\psi(x,t+s_j) = \psi''(x,t+s_j) - m^2\psi(x,t+s_j) + \delta(x) f(\psi(0,t+s_j))$$

$$\psi(\cdot,t+s_j) \xrightarrow{H^1_{\rm loc}(\mathbb{R})} \beta(\cdot,t) \Longrightarrow \psi(0,t+s_j) \to \beta(0,t). \qquad \square$$

Summary:

i) $\ddot\beta(x,t) = \beta''(x,t) - m^2\beta(x,t) + \delta(x) f(\gamma(t)), \quad \gamma(t) = \beta(0,t)$

ii) $\mathrm{supp}\ \tilde\beta(x,\cdot) = \mathrm{supp}\ \tilde\gamma \subset [-m,m], \quad \forall x \in \mathbb{R}.$

We will deduce: $\beta(x,t) = \psi_*(x) e^{i\omega_* t}$

Spectral Inclusion

Recall **condition C3 of Strict Nonlinearity:**

$$U(\psi) = u(|\psi|) = \Sigma_0^N u_j |\psi|^{2j}, \quad u_N > 0, \ N \geq 2. \quad \text{(SNL)}$$

Then $f(\psi) = a(|\psi|)\psi$, and

$$f_0(t) := f(\gamma(t)) = a(|\gamma(t)|)\gamma(t) = A(t)\gamma(t)$$

Lemma $\mathrm{supp}\ \tilde{A} * \tilde{\gamma} \subset \mathrm{supp}\ \tilde{\gamma}$.

Proof:

$$-\omega^2 \tilde{\beta}(x,\omega) = \tilde{\beta}''(x,\omega) - m^2 \tilde{\beta}(x,\omega) + \delta(x)\tilde{f}_0(\omega)$$

$$-\omega^2 \tilde{\beta}(x,\omega) = \tilde{\beta}''(x,\omega) - m^2 \tilde{\beta}(x,\omega) + \delta(x)[\tilde{A} * \tilde{\gamma}](\omega)$$

Spectral Identity: $\mathrm{supp}\ \tilde{\beta}(x,\cdot) = \mathrm{supp}\ \tilde{\gamma}, \ \forall x \in \mathbb{R}.$ □

Titchmarsh convolution theorem

Theorem (Titchmarsh, 1926).
Let $f(\omega)$ and $g(\omega)$ be distributions of $\omega \in \mathbb{R}$ with compact supports. Then $[\mathrm{supp}\; f * g] = [\mathrm{supp}\; f] + [\mathrm{supp}\; g]$, where $[X]$ denotes the convex hull of a subset $X \subset \mathbb{R}$.

$[\mathrm{supp}\; \tilde{A} * \tilde{\gamma}] = [\mathrm{supp}\; \tilde{A}] + [\mathrm{supp}\; \tilde{\gamma}] \subset [\mathrm{supp}\; \tilde{\gamma}] \Longrightarrow [\mathrm{supp}\; \tilde{A}] = \{0\}$,
$\tilde{A}(\omega) = C\delta(\omega) \Longrightarrow a(|\gamma(t)|) \equiv C_1$. Hence, $|\gamma(t)| \equiv C_2$ by (SNC).

Corollary $\mathrm{supp}\; \tilde{\gamma} = \{\omega_*\}$. **Proof:** $\gamma(t)\overline{\gamma}(t) = C_2^2$, hence
$\tilde{\gamma} * \tilde{\overline{\gamma}} = C_3\delta(x)$. But $\tilde{\overline{\gamma}}(\omega) = \overline{\tilde{\gamma}(-\omega)} \Longrightarrow \mathrm{supp}\; \tilde{\overline{\gamma}} = -\mathrm{supp}\; \tilde{\gamma}$.
Hence, $[\mathrm{supp}\; \tilde{\gamma}] - [\mathrm{supp}\; \tilde{\gamma}] = \{0\}$ by Titchmarsh Theorem. □

Remark Titchmarsh Theorem is not applicable to Schrödinger equation since the **'spectral gap' $(-\infty, 0)$ is infinite**.

Generalizations: The global attraction is proved

i) in 2006-2011 together with A. Comech for Eqns

$$\ddot\psi(x,t) = \psi''(x,t) - m^2\psi + \sum_{k=1}^{N} \delta(x-x_k) f_k(\psi(x_k,t)), \quad x \in \mathbb{R}$$

$$\ddot\psi(x,t) = \Delta\psi(x,t) - m^2\psi + \rho(x) f(\langle\psi(\cdot,t),\rho\rangle), \qquad x \in \mathbb{R}^n$$

$$i\dot\psi(x,t) = (\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m)\psi + \rho(x) f(\langle\psi(\cdot,t),\rho\rangle), \qquad x \in \mathbb{R}^n$$

The Wiener condition (Fermi Golden Rule) : $\hat\rho(k) \neq 0, \ k \in \mathbb{R}^n$

ii) in 2012 by A. Comech for discrete in space and time nonlinear KG.

iii) in 2016-2017 by E. Kopylova for 3D nonlinear wave and KG Eqns with concentrated nonlinearities.

VI. Open problems: the global attraction for

i) the Klein-Gordon equation (KG) and *fixed solitary waves*

$\psi(x,t) \sim \psi_{\omega_\pm}(x) e^{i\omega_\pm t}, \quad t \to \pm\infty$

ii) the Schrödinger equation $\quad i\dot{\psi}(x,t) = \psi''(x,t) + \delta(x) f(\psi(0,t))$

iii) **relativistic** nonlinear KG Eqn

$\ddot{\psi}(x,t) = \psi''(x,t) - m^2\psi(x,t) + f(\psi(x,t)), \quad f(\psi) = -\nabla_\psi U(\psi)$

iv) coupled Maxwell-Schrödinger equations

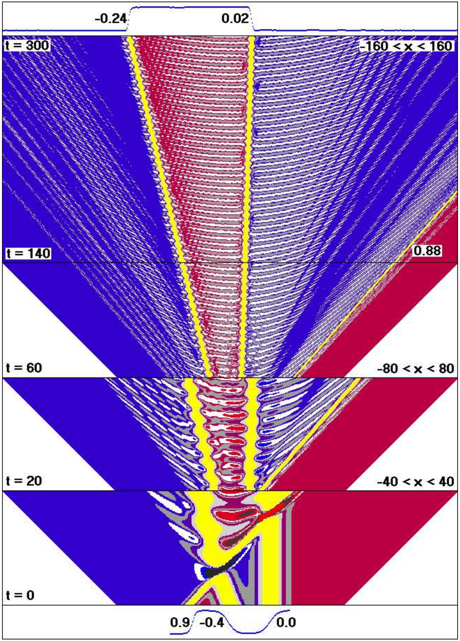
$i\dot{\psi}(x,t) = \frac{1}{2}[-i\nabla + \mathbf{A}(x,t) + \mathbf{A}^{\rm ext}(x,t)]^2\psi + [A_0(x,t) + A_0^{\rm ext}(x)]\psi$

$\Box A_\alpha(x,t) = 4\pi J_\alpha(x,t), \qquad \alpha = 0,1,2,3$

v) coupled Maxwell-Dirac equations

$\sum_{\alpha=0}^3 \gamma^\alpha [i\nabla_\alpha - A_\alpha(x,t) - A_\alpha^{\rm ext}(x,t)]\psi(x,t) = m\psi(x,t)$

$\Box\, A_\alpha(x,t) = J_\alpha(x,t) := \overline{\psi(x,t)}\gamma^0\gamma_\alpha\psi(x,t), \quad \alpha = 0,1,2,3$



-0.24
0.02
t = 300
-160 < x < 160
t = 140
0.88
t = 60
-80 < x < 80
t = 20
-40 < x < 40
t = 0
0.9
-0.4
0.0

Adiabatic effective dynamics of relativistic solitons and mass-energy equivalence

$$\ddot{\psi} = \Delta\psi - m^2\psi + f(\psi(x,t)) - V(x)\psi(x,t)), \qquad f(\psi) = -\nabla U(\psi)$$

$$H_0(\psi,\pi) = \frac{1}{2}\int[|\pi|^2 + |\nabla\psi|^2 + m^2|\psi|^2]dx + \int U(\psi(x))dx,$$

$$H = H_0 + \int V(x)|\psi(x)|^2 dx$$

$V = 0:$

i) Momentum conservation $P(t) := -\int \dot{\psi}(x,t)\nabla\psi(x,t)dx = \mathrm{const}$

ii) Energy conservation $E(t) := H_0(\psi(\cdot,t),\pi(\cdot,t)) = \mathrm{const}$

iii) Solitons $\psi_{v,Q} := \psi_v(x - vt - Q)$, $\pi_{v,Q} := v\nabla\psi_v(x - vt - Q)$,

Existence of solitons: W. Strauss 1977, H.Beresticky & P.-L.Lions 1983

Definition. $H_0^{\rm eff}(P_v) = H_0(\psi_{v,Q}\,,\,\pi_{v,Q}),\quad P_v := -\int \pi_{v,Q}(x,t)\nabla\psi_{v,Q}(x,t)dx$

Effective dynamics: $\dot{Q} = \nabla_P H_0^{\rm eff}(P),\quad \dot{P} = -\nabla_Q V(Q)$

Problem I. Let $\sup_{x\in\mathbb{R}^3}|\nabla V(x)| \le \varepsilon,$

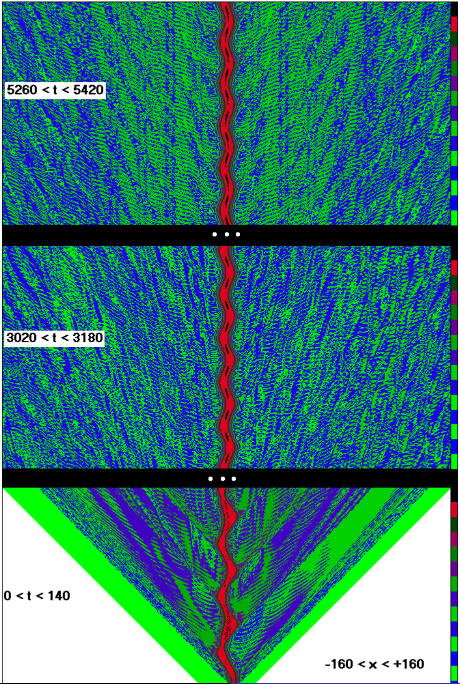
$$\psi(x,0) = \psi_{v_0,Q_0}\,,\quad \pi(x,0) = \pi_{v_0,Q_0}$$

$$Q(0) = Q_0\,,\qquad P(0) = \frac{v_0}{\sqrt{1+v_0^2}}$$

Prove: $\psi(x-Q(t),t) \sim \psi_{v(t)}(x-Q(t))$ for $|t| \le \varepsilon^{-1},\quad v(t) := \dot{Q}(t)$

Problem II. Prove: $H_0^{\rm eff}(P_v) \sim \dfrac{P^2}{2m_e},\quad P \to 0,$ where

$m_e = H_0^{\rm eff}(P_0)$ (Einstein mass-energy equivalence $E = mc^2$)


5260 < t < 5420
3020 < t < 3180
0 < t < 140
-160 < x < +160

VII. General conjecture for G-invariant equations

The results on the attraction correspond to the symmetry group of Eqns:

I. Attraction to stationary states: $G = e$

II. Attraction to solitons: $G = \mathbb{R}^n$

III. Attraction to stationary orbits: $G = U(1)$

G-Conjecture: For 'generic' G-invariant Hamilton nonlinear PDEs

$$\psi(t) \sim e^{g_\pm t}\psi_\pm, \qquad t \to \pm\infty$$

for each finite energy solution, where $g_\pm \in \mathfrak{g}$.

Experimental confirmation **symmetry group $\leftrightarrow$ asymptotic states**

Gell-Mann, Ne'eman 1961: Dynkin scheme of $su(3)$ with 8 generators corresponds to 8 baryons ('eightfold way').

7 baryons were known in 1961, 8-th Ω^--hyperon was discovered in 1964.

VIII. Comparison with attractors of dissipative PDEs

For dissipative systems since 1970: Foias, Temam, Vishik, Titi, ...

Navier-Stokes eqns, reaction-diffusion eqns, nonlinear parabolic eqns, damped wave eqns, etc

i) The convergence holds in **bounded and unbounded regions**;

ii) In **global energy norm**;

iii) For $t \to +\infty$ **only**.

For Hamiltonian PDEs since 1990: K, Spohn, Comech, Kopylova, ...

i) The convergence holds **only** in unbounded regions;

ii) In **local energy seminorms**;

iii) For $t \to \pm\infty$.

THANK YOU !