Spectral properties of 1D-Dirac operators with point interactions.

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Content

Spectral analysis of the GS-operators $D_{X,\alpha}(Q)$ and $D_{X,\beta}(Q)$ consists (at least partially) of the following problems:

(a) Finding of self-adjointness criteria for $D_{X,\alpha}(Q)$ and $D_{X,\beta}(Q)$.

(b) Discreteness of the spectra of the operators $D_{X,\alpha}(Q)$ and $D_{X,\beta}(Q)$.

(c) Characterization of continuous, absolutely continuous, and singular parts of the spectra of the operators $D_{X,\alpha}(Q)$ and $D_{X,\beta}(Q)$.

(d) Resolvent comparability of the operators $D_{X,\alpha^{(1)}}(Q)$ and $D_{X,\alpha^{(2)}}(Q)$ with $\alpha^{(1)} \neq \alpha^{(2)}$ i.e. finding conditions for the inclusion $(D_{X,\alpha^{(1)}}(Q) - i)^{-1} - (D_{X,\alpha^{(2)}}(Q) - i)^{-1} \in \mathfrak{S}_p(\mathfrak{H})$, to be valid, where $\mathfrak{S}_p(\mathfrak{H})$ denotes the Neumann-Schatten ideal.

1-D Dirac operators with point interactions

We consider the Dirac differential expression

$$D \equiv D^c := -i\,c\,\frac{d}{dx} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{c^2}{2} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv \begin{pmatrix} c^2/2 & -i\,c\,\frac{d}{dx} \\ -i\,c\,\frac{d}{dx} & -c^2/2 \end{pmatrix}. \tag{1}$$

acting on $\mathbb{C}^2$-valued functions of a real variable. Here

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2}$$

are the Pauli matrices in $\mathbb{C}^2$ and $c > 0$ denotes the velocity of light. We set

$$k(z) := c^{-1}\sqrt{z^2 - \left(c^2/2\right)^2}, \qquad z \in \mathbb{C}, \tag{3}$$

where the branch of the multifunction $\sqrt{\cdot}$ is selected such that $k(x) > 0$ for $x > c^2/2$. It is easily seen that $k(\cdot)$ is holomorphic in $\mathbb{C}$ with two cuts along the half-lines $\left(-\infty, -c^2/2\right]$ and $\left[c^2/2, \infty\right)$ and $k(\overline{z}) = -\overline{k(z)}$.

Let $\alpha := \{\alpha_n\}_{n=1}^{\infty} \subset \mathbb{R} \cup \{+\infty\}$ and $\beta := \{\beta_n\}_{n=1}^{\infty} \subset \mathbb{R} \cup \{+\infty\}$. Let $X = \{x_n\}_{n=1}^{\infty}$ be a discrete subset of the interval $\mathcal{I} = (a, b)$, $x_{n-1} < x_n,\ n \in \mathbb{N}$, with the accumulative point b. Assuming that $-\infty < a < b \le +\infty$ and $I = \mathbb{N}$, define the op-s $D_{X,\alpha}$ and $D_{X,\beta}$ (realizations of D) to be the closures of the op-s $D^0_{X,\alpha} = D$ and $D^0_{X,\beta} = D$, being restrictions of D to the domains

$$\operatorname{dom}(D^0_{X,\alpha}) = \Big\{ f \in W^{1,2}_{comp}(\mathcal{I}\backslash X) \otimes \mathbb{C}^2 : f_1 \in AC_{\rm loc}(\mathcal{I}),\ f_2 \in AC_{\rm loc}(\mathcal{I}\backslash X); \\ f_2(a+) = 0\,, \quad f_2(x_n+) - f_2(x_n-) = -\frac{i\alpha_n}{c} f_1(x_n),\ n \in \mathbb{N} \Big\}, \tag{4}$$

$$\operatorname{dom}(D^0_{X,\beta}) = \Big\{ f \in W^{1,2}_{comp}(\mathcal{I}\backslash X) \otimes \mathbb{C}^2 : f_1 \in AC_{\rm loc}(\mathcal{I}\backslash X),\ f_2 \in AC_{\rm loc}(\mathcal{I}); \\ f_2(a+) = 0\,, \quad f_1(x_n+) - f_1(x_n-) = i\beta_n c f_2(x_n),\ n \in \mathbb{N} \Big\}, \tag{5}$$

respectively, i.e. $D_{X,\alpha} = \overline{D^0_{X,\alpha}}$ and $D_{X,\beta} = \overline{D^0_{X,\beta}}$.

Boundary triplets for the operator D_X^*

To treat realizations $D_{X,\alpha}$ and $D_{X,\beta}$ in the framework of extensions theory we introduce the minimal operators D_n generated in $L^2[x_{n-1}, x_n] \otimes \mathbb{C}^2$ by the expression (1),

$$D_n = D \upharpoonright \operatorname{dom}(D_n), \quad \operatorname{dom}(D_n) = W_0^{1,2}[x_{n-1}, x_n] \otimes \mathbb{C}^2. \tag{6}$$

We also put $d_n := x_n - x_{n-1} > 0$.
The triplet $\widetilde{\Pi}^{(n)} = \{\mathbb{C}^2, \widetilde{\Gamma}_0^{(n)}, \widetilde{\Gamma}_1^{(n)}\}$, where

$$\widetilde{\Gamma}_0^{(n)} f := \begin{pmatrix} f_1(x_{n-1}+) \\ i\,c\,f_2(x_n-) \end{pmatrix}, \qquad \widetilde{\Gamma}_1^{(n)} f := \begin{pmatrix} i\,c\,f_2(x_{n-1}+) \\ f_1(x_n-) \end{pmatrix}, \tag{7}$$

forms a boundary triplet for D_n^*.
To treat the operators $D_{X,\alpha}$ and $D_{X,\beta}$ as extensions of the minimal operator $D_X := \bigoplus_{n=1}^{\infty} D_n$ we construct a boundary triplet for the operator $D_X^* := \bigoplus_{n=1}^{\infty} D_n^*$.

Theorem 1

Let $X=\{x_n\}_{n=1}^{\infty}$ be as above and $d^*(X)<+\infty$. Define the mappings

$$\Gamma_j^{(n)}: W^{1,2}[x_{n-1},x_n]\otimes\mathbb{C}^2\to\mathbb{C}^2\,,\quad n\in\mathbb{N}\,,\quad j\in\{0,1\}\,,$$

by setting

$$\Gamma_0^{(n)}f := \begin{pmatrix} d_n^{1/2} f_1(x_{n-1}+) \\ i\,c\,d_n^{3/2}\sqrt{1+\frac{1}{c^2 d_n^2}}\, f_2(x_n-) \end{pmatrix}, \tag{8}$$

$$\Gamma_1^{(n)}f := \begin{pmatrix} i\,c\,d_n^{-1/2}\,(f_2(x_{n-1}+)-f_2(x_n-)) \\ d_n^{-3/2}\left(1+\frac{1}{c^2\,d_n^2}\right)^{-1/2}(f_1(x_n-)-f_1(x_{n-1}+)-i\,c\,d_n\,f_2(x_n-)) \end{pmatrix}. \tag{9}$$

Then:

(i) $\Pi^{(n)}=\{\mathbb{C}^2,\Gamma_0^{(n)},\Gamma_1^{(n)}\}$ is a boundary triplet for D_n^*, $n\in\mathbb{N}$.

(ii) The direct sum $\Pi:=\bigoplus_{n=1}^{\infty}\Pi^{(n)}=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ with $\mathcal{H}=l^2(\mathbb{N},\mathbb{C}^2)$ and $\Gamma_j=\bigoplus_{n=1}^{\infty}\Gamma_j^{(n)}$, $j\in\{0,1\}$, is a boundary triplet for the operator $D_X^*=\bigoplus_{n=1}^{\infty}D_n^*$.

Connection with Jacobi matrices

Consider the Jacobi matrix

$$B_{X,\alpha} = \begin{pmatrix} 0 & -\frac{\nu(d_1)}{d_1^2} & 0 & 0 & 0 & \dots \\ -\frac{\nu(d_1)}{d_1^2} & -\frac{\nu(d_1)}{d_1^2} & \frac{\nu(d_1)}{d_1^{3/2}d_2^{1/2}} & 0 & 0 & \dots \\ 0 & \frac{\nu(d_1)}{d_1^{3/2}d_2^{1/2}} & \frac{\alpha_1}{d_2} & -\frac{\nu(d_2)}{d_2^2} & 0 & \dots \\ 0 & 0 & -\frac{\nu(d_2)}{d_2^2} & -\frac{\nu(d_2)}{d_2^2} & \frac{\nu(d_2)}{d_2^{3/2}d_3^{1/2}} & \dots \\ 0 & 0 & 0 & \frac{\nu(d_2)}{d_2^{3/2}d_3^{1/2}} & \frac{\alpha_2}{d_3} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \tag{10}$$

where

$$\nu(x) := \frac{1}{\sqrt{1+\frac{1}{c^2x^2}}}. \tag{11}$$

$$B_{X,\beta} := \begin{pmatrix} 0 & -\frac{\nu(d_1)}{d_1^2} & 0 & 0 & \dots \\ -\frac{\nu(d_1)}{d_1^2} & -\frac{\nu^2(d_1)}{d_1^3}(\beta_1 + d_1) & \frac{\nu(d_1)}{d_1^{3/2}\, d_2{}^{1/2}} & 0 & \dots \\ 0 & \frac{\nu(d_1)}{d_1^{3/2} d_2{}^{1/2}} & 0 & -\frac{\nu(d_2)}{d_2^2} & \dots \\ 0 & 0 & -\frac{\nu(d_2)}{d_2^2} & -\frac{\nu^2(d_2)}{d_2^3}(\beta_2 + d_2) & \dots \\ 0 & 0 & 0 & \frac{\nu(d_2)}{d_2^{3/2} d_3^{1/2}} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \tag{12}$$

Proposition 1 (basic lemma)

Let $D_X = \bigoplus_{n=1}^{\infty} D_n$ be the minimal Dirac operator in $L^2(\mathcal{I}, \mathbb{C}^{2p})$. Let also $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be the boundary triplet for $D_X^* = \bigoplus_{n=1}^{\infty} D_n^*$ constructed in Theorem 1 and let $B_{X,\alpha}$ ($B_{X,\beta}$) be the minimal Jacobi operator associated in $l^2(\mathbb{N}, \mathbb{C}^{2p})$ with the matrix (10) ((12)). Then the boundary operator corresponding to the GS-realization $D_{X,\alpha}$ ($D_{X,\beta}$) in the triplet Π, is the Jacobi operator $B_{X,\alpha}$ ($B_{X,\beta}$), i.e.

$$D_{X,\alpha} = D_{B_{X,\alpha}} = D_X^* \restriction \mathrm{dom}(D_{B_{X,\alpha}}),$$

$$\mathrm{dom}(D_{B_{X,\alpha}}) = \{f \in W^{1,2}(\mathcal{I} \setminus X) \otimes \mathbb{C}^{2p} : \Gamma_1 f = B_{X,\alpha} \Gamma_0 f\},$$

$$D_{X,\beta} = D_{B_{X,\beta}} := D_X^* \restriction \mathrm{dom}(D_{B_{X,\beta}}),$$

$$\mathrm{dom}(D_{B_{X,\beta}}) := \{f \in W^{1,2}(\mathcal{I} \setminus X) \otimes \mathbb{C}^{2p} : \ \Gamma_1 f = B_{X,\beta} \Gamma_0 f\}.$$

Theorem 3

Let $\alpha := \{\alpha_n\}_{n=1}^{\infty} \subset \mathbb{R} \cup \{+\infty\}$ and let $B_{X,\alpha}$ be the minimal Jacobi operator associated with the Jacobi matrix (10). Then:

(*i*) $\mathrm{n}_\pm(D_{X,\alpha}) = \mathrm{n}_\pm(B_{X,\alpha})$, hence the GS-operator $D_{X,\alpha}$ has equal deficiency indices and $\mathrm{n}_+(D_{X,\alpha}) = \mathrm{n}_-(D_{X,\alpha}) \le 1$.

In particular, $D_{X,\alpha}$ is self-adjoint if and only if $B_{X,\alpha}$ is.

Additionally, let $D_{X,\alpha} = D_{X,\alpha}^*$. Then:

(*ii*) The operator $D_{X,\alpha}$ has a discrete spectrum, if and only if $\lim_{n\to+\infty} d_n = 0$ and $B_{X,\alpha}$ has discrete spectrum.

(*iii*) Let $\widetilde{\alpha} := \{\widetilde{\alpha}_n\}_{n\in\mathbb{N}} (\subset \mathbb{C}^{2\times 2})$, $\widetilde{\alpha}_n = (\widetilde{\alpha}_n)^*$. Let also $B_{X,\widetilde{\alpha}}$ be Jacobi operator associated in $\mathcal{H} = l^2(\mathbb{N}) \otimes \mathbb{C}^{2\times 2}$ with matrix (10), with α replaced by $\widetilde{\alpha}$. Then the following equi-ce holds

$$(D_{X,\alpha}-i)^{-1}-(D_{X,\widetilde{\alpha}}-i)^{-1} \in \mathfrak{S}_p(\mathfrak{H}) \iff (B_{X,\alpha}-i)^{-1}-(B_{X,\widetilde{\alpha}}-i)^{-1} \in \mathfrak{S}_p(\mathcal{H}).$$

Self-adjointness

Combining Theorem 3(i) with the Carleman test, we obtain the following result.

Proposition 2

Let $\mathcal{I}$ be an infinite interval, i.e. either $\mathcal{I} = \mathbb{R}_{\pm}$ or $\mathcal{I} = \mathbb{R}$. Then the GS-realization $D_{X,\alpha}$ is self-adjoint for any $\alpha = \{\alpha_n\}_{n=1}^{\infty} \subset \mathbb{R} \cup \infty$.

Proposition 3

Assume that $|\mathcal{I}| < \infty$. Then the *GS*-realization $D_{X,\alpha}$ in $L^2(\mathcal{I}, \mathbb{C}^2)$ is selfadjoint provided that

$$\sum_{n\in\mathbb{N}} \sqrt{d_n d_{n+1}}\, |\alpha_n| = +\infty \,. \tag{13}$$

Self-adjointness

Example

Let $\mathcal{I} := (0,1)$ and let the sequence $X = \{x_n\}_{n=1}^{\infty} \subset (0,1)$ be given by $x_n = 1 - 1/2^n$, so that $d_n = 1/2^n$. Let also $\alpha = \{\alpha_n\}_1^{\infty}$ be given by $\alpha_n = (-3)2^n + 1,\ n \in \mathbb{N}$. By Proposition 3, the *GS*-operator $D_{X,\alpha}$ on $L^2(0,1) \otimes \mathbb{C}^2$ is selfadjoint since the series $\sum_{n=1}^{\infty} \alpha_n/2^n$ diverges.
On the other hand, it is easily seen that

$$\{d_n\}_1^{\infty} \in l^2\,, \quad d_{n-1}d_{n+1} = \frac{1}{2^{2n}} = d_n^2$$

and

$$\sum_{n=1}^{\infty} d_{n+1}\left|\alpha_n + \frac{1}{d_n} + \frac{1}{d_{n+1}}\right| = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2}.$$

Discreteness of spectrum

Consider differential expression

$$D(Q) := D^c(Q) := -i\,c\,\frac{d}{dx}\otimes\sigma_1+\frac{c^2}{2}\otimes\sigma_3+Q(x), \qquad Q(x)=Q(x)^*, \tag{14}$$

and denote by $D_X(Q) := D^c_X(Q)$ the minimal operator associated on $\mathcal{I}\setminus X$ with the expression $D^c(Q)$. One has

$$D_X(Q) = D(Q)\restriction \mathrm{dom}(D_X(Q)),$$

$$\mathrm{dom}(D_X(Q)) = W_0^{1,2}\left(\mathcal{I}\setminus X,\mathbb{C}^{2p}\right) = \bigoplus_{n=1}^{\infty} W_0^{1,2}\left([x_{n-1},x_n],\mathbb{C}^{2p}\right). \tag{15}$$

Proposition 4

Let $X = \{x_n\}_1^\infty (\subset \mathbb{R}_+)$, $\alpha = \{\alpha_j\}_1^\infty \subset \mathbb{R}$ and let $Q(\cdot) = Q^*(\cdot) \in L^2_{\rm loc}(\mathbb{R}_+) \otimes \mathbb{C}^{2\times 2}$ be strongly subordinated to $D^c_{X,\alpha} = D^c_{X,\alpha}(0)$. Assume also that $\lim_{n\to\infty} d_n = 0$ and

$$\lim_{n\to\infty} \frac{|\alpha_n|}{d_n} = \infty \qquad \text{and} \qquad \lim_{n\to\infty} \frac{c}{\alpha_n} > -\frac{1}{4}. \tag{16}$$

Then the *GS*-operator $D^c_{X,\alpha}(Q)$ on the half-line $\mathbb{R}_+$ has discrete spectrum.

Continuous spectrum

let $D_{X,\alpha}(Q) := D_{X,\alpha} + Q$ be the GS realization of $D(Q)$. If $\alpha := \mathbf{0} = \{0\}_1^\infty$ is a zero sequence we set $D_N(Q) := D_{X,\mathbf{0}}(Q)$ and note that $D_N(Q)$, the Neumann realization of $D(Q)$, is given by

$$D_N(Q) = D(Q) \upharpoonright \operatorname{dom}(D_N(Q)),$$

$$\operatorname{dom}(D_N(Q)) = \operatorname{dom}(D_N) = \{f \in W^{1,2}(\mathbb{R}_+) \otimes \mathbb{C}^2 : \ f_2(x_0+) = 0\}.$$

Theorem 4

Assume that $Q \in L^\infty(\mathbb{R}_+) \otimes \mathbb{C}^{2\times 2}$, $Q(x) = Q^*(x)$ for a.e. $x \in \mathbb{R}_+$, and $\alpha = \{\alpha_n\}_1^\infty \subset \mathbb{R}$. Then the following holds
(i) If $\{\alpha_n/d_{n+1}\}_1^\infty \in c_0(\mathbb{N})$, then

$$\sigma_{\text{ess}}\big(D_{X,\alpha}(Q)\big) = \sigma_{\text{ess}}\big(D_N(Q)\big). \tag{17}$$

Moreover, if in addition, $Q(x) \to 0$ as $x \to \infty$, then

$$\sigma_{\text{ess}}\big(D_{X,\alpha}(Q)\big) = \sigma_{\text{ess}}\big(D_N(Q)\big) = \mathbb{R} \setminus (-c^2/2, c^2/2). \tag{18}$$

(ii) If $\{\alpha_n/d_{n+1}\}_1^\infty \in l^1(\mathbb{N})$, then

$$\sigma_{\rm ac}\big(D_{X,\alpha}(Q)\big) = \sigma_{\rm ac}\big(D_N(Q)\big). \tag{19}$$

Moreover, if additionally, $Q \in L^1(\mathbb{R}_+) \otimes \mathbb{C}^{2\times 2}$, then

$$\sigma_{\rm ac}\big(D_{X,\alpha}(Q)\big) = \sigma_{\rm ac}\big(D_N(Q)\big) = \mathbb{R} \setminus (-c^2/2, c^2/2). \tag{20}$$

(iii) If condition $\sigma_{ac}(D_{X,\alpha}) = \emptyset$ is satisfied, then the spectrum of $D_{X,\alpha}(Q)$ is purely singular, i.e.

$$\sigma_{\rm ac}\big(D_{X,\alpha}(Q)\big) = \emptyset. \tag{21}$$

(iv) Assume in addition that $d_*(X) > 0$. Then the above assumptions can be replaced by

$$\{\alpha_n\}_1^\infty \in c_0(\mathbb{N}), \qquad \{\alpha_n\}_1^\infty \in l^1(\mathbb{N}) \qquad \text{and} \qquad \limsup_{n\to\infty} |\alpha_n| = \infty,$$

respectively.

Non-relativistic limit of Gesztesy-Šeba operators

Theorem 5

Let $X = \{x_n\}_1^\infty (\subset \mathbb{R}_+)$ be a discrete set and $\alpha = \{\alpha_n\}_1^\infty$, $\beta = \{\beta_n\}_1^\infty \subset \mathbb{R}$.

(*i*) Assume that $\mathcal{I} = \mathbb{R}_+$ and $\mathrm{H}_{X,\alpha}$ is selfadjoint. Then

$$s - \lim_{c\to+\infty} \left(D^c_{X,\alpha} - (z + c^2/2)\right)^{-1} = (\mathrm{H}_{X,\alpha} - z)^{-1} \bigotimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{22}$$

In particular, (22) holds provided that $\mathrm{H}_{X,\alpha}$ is semibounded below.

(*ii*) Assume that $\mathcal{I} = \mathbb{R}_+$. Then the operators $D^c_{X,\beta}$, $c < \infty$, and $\mathrm{H}_{X,\beta}$ are selfadjoint and the following relation holds

$$s - \lim_{c\to+\infty} \left(D^c_{X,\beta} - (z + c^2/2)\right)^{-1} = (\mathrm{H}_{X,\beta} - z)^{-1} \bigotimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{23}$$

(*iii*) Assume that $|\mathcal{I}| < \infty$, i.e. $\mathcal{I} = (0, b)$. Assume also that

$$\sum_{n=1}^{\infty} |\beta_n| \sqrt{d_n d_{n+1}} = +\infty \qquad \text{and} \qquad \sum_{n=1}^{\infty} \Big(d_{n+1} \Big| \sum_{i=1}^{n} (\beta_i + d_i) \Big|^2 \Big) = \infty. \tag{24}$$

Then the relation (23) holds.

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Thank you!