1 3D Klein-Gordon equation

$$i\dot{\Psi}(t) = \mathcal{H}\Psi(t), \quad \mathcal{H} = i\begin{pmatrix} 0 & 1 \\ \Delta - m^2 - V & 0 \end{pmatrix}$$

$$(\mathbf{1}) \quad |V(x)| + |\nabla V(x)| \le C(1+|x|)^{-\beta}, \quad \beta > 3, \quad x \in \mathbb{R}^3$$

$(\mathbf{2})$ $\lambda = 0$ is neither eigenvalue nor resonance of $H = -\Delta + V(x)$

Theorem 1.1 *([KK]) Let* $\Psi(0) \in E_\sigma$, $\sigma > 5/2$. *Then*

$$\|\mathcal{P}_c\Psi(t)\|_{E_{-\sigma}} = \mathcal{O}(|t|^{-3/2}), \quad t \to \pm\infty.$$

[KK] A. Komech, E. Kopylova, Weighted energy decay for 3D Klein-Gordon equation, *J. Diff. Eqns.* **248** (2010), no. 3, 501-520.

Free 3D Klein-Gordon equation

$$\Psi(t) = \frac{1}{2\pi i} \int\limits_{\Gamma} e^{-i\omega t} \Big[R_{KG}(\omega + i0) - R_{KG}(\omega - i0) \Big] \Psi(0) d\omega, \quad \Psi = \begin{pmatrix} \psi \\ \dot{\psi} \end{pmatrix}$$

$$R_{KG}(\omega, x - y) = \frac{e^{i\sqrt{\omega^2 - m^2}|x-y|}}{4\pi |x - y|} \begin{pmatrix} \omega & i \\ -i\omega^2 & \omega \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -i\delta(x - y) & 0 \end{pmatrix}$$

$$\Gamma = (-\infty, -m] \cup [m, \infty)$$

We have

$$\begin{aligned} R_{\mathrm{KG}}(\omega \pm i0) &= \mathcal{O}(1), \quad \omega \to \pm m \\ R'_{\mathrm{KG}}(\omega \pm i0) &= \mathcal{O}((\omega \mp m)^{-1/2}), \quad \omega \to \pm m \\ R''_{\mathrm{KG}}(\omega \pm i0) &= \mathcal{O}((\omega \mp m)^{-3/2}), \quad \omega \to \pm m \end{aligned}$$

in $E_\sigma \to E_{-\sigma}$ with $\sigma > 5/2$.

Then the contribution of low frequencies decays $\sim t^{-3/2}$ as $t \to \infty$ in $E_\sigma \to E_{-\sigma}$ with $\sigma > 5/2$.

High frequencies Free Schrödinger equation.

$$R_S(\omega) = \frac{e^{i\sqrt{\omega}|x-y|}}{4\pi|x-y|}$$

$$\psi_h(t) = \frac{1}{2\pi i} \int\limits_{[0,\infty)} \zeta(\omega) e^{-i\omega t} [R_S(\omega + i0) - R_S(\omega - i0)] \psi(0) d\omega$$

$$\zeta(\omega) \in C^\infty, \quad \zeta(\omega) = 0, \quad \text{for } \omega \le 1$$

We plan to apply integration by parts. For this we will use the result of Agmon [A]: For any $\sigma > 1/2$ the following asymptotics hold

$$\|R_{\rm S}(\omega)\|_{H^s_\sigma \to H^{s+k}_{-\sigma}} = \mathcal{O}(\omega^{-\frac{1-k}{2}}), \quad \omega \to \infty, \quad \omega \in \mathbb{C} \setminus [0,\infty), \quad k = -1, 0, 1, 2$$

Hence

$$\|R_{\rm S}(\omega \pm i0)\|_{L^2_\sigma \to L^2_{-\sigma}} = \mathcal{O}(\omega^{-\frac{1}{2}}), \ \omega \to \infty, \quad \sigma > 1/2$$

[A] S. Agmon S, Spectral properties of Schrödinger operator and scattering theory, *Ann.Scuola Norm. Sup.Pisa*, Ser. IV **2**, 151-218 (1975).

Theorem 1.2 *For $j = 1, 2, ...$ and any $\sigma > j + 1/2$*

$$\|R_S^{(j)}(\omega)\|_{H^s_\sigma \to H^{s+k}_{-\sigma}} = \mathcal{O}(|\omega|^{-\frac{1-k+j}{2}}),\ \omega \to \infty,\ \omega \in \mathbb{C}\setminus[0,\infty),\quad k=-1,0,1,2$$

In [KK] we give a complete proof of this theorem refining the arguments in the proof of Theorem A.1 from Appendix A in [A].

For $k = 0$ amd $j = 0, 1, 2$, we get

$$\|R_{\rm S}(\omega \pm i0)\|_{L^2_\sigma \to L^2_{-\sigma}} = \mathcal{O}(\omega^{-\frac{1}{2}}),\ \omega \to \infty, \quad \sigma > 1/2$$

$$\|R'_{\rm S}(\omega \pm i0)\|_{L^2_\sigma \to L^2_{-\sigma}} = \mathcal{O}(\omega^{-1}),\ \omega \to \infty, \quad \sigma > 3/2$$

$$\|R''_{\rm S}(\omega \pm i0)\|_{L^2_\sigma \to L^2_{-\sigma}} = \mathcal{O}(\omega^{-\frac{3}{2}}),\ \omega \to \infty, \quad \sigma > 5/2$$

[KK] A. Komech, E. Kopylova, Weighted energy decay for 3D Klein-Gordon equation, *J. Differential Equations* **248** (2010), no. 3, 501-520.

Now two integration by parts in

$$\psi_h(t) = \frac{1}{2\pi i} \int\limits_{[0,\infty)} \zeta(\omega) e^{-i\omega t} [R_S(\omega + i0) - R_S(\omega - i0)] \psi(0) d\omega$$

gives that for $\psi_0 \in L^2_\sigma$ with $\sigma > 5/2$

$$\|\psi_h(t)\|_{L^2_{-\sigma}} = \mathcal{O}(t^{-2}), \quad t \to \infty$$

High frequencies Free Klein-Gordon equation.
Recall that for $k = -1, 0, 1$ and $\sigma > 1/2$

$$\|R_{\rm S}(\omega)\|_{H^s_\sigma \to H^{s+k}_{-\sigma}} = \mathcal{O}(\omega^{-\frac{1-k}{2}}), \quad \omega \to \infty, \quad \omega \in \mathbb{C} \setminus [0, \infty)$$

We need asymptotics for

$$R_{\rm KG}(\omega) = R_{\rm S}(\omega^2 - m^2) \left(\begin{array}{cc} \omega & i \\ -i\omega^2 & \omega \end{array} \right), \quad \omega \to \infty, \quad \omega \in \mathbb{C} \setminus [0, \infty)$$

in $H^1_\sigma \oplus H^0_\sigma \to H^1_{-\sigma} \oplus H^0_{-\sigma}$. We have

$$\|R^{11}_{\rm KG}(\omega)\|_{H^1_\sigma \to H^1_{-\sigma}}, \ \|R^{22}_{\rm KG}(\omega)\|_{H^0_\sigma \to H^0_{-\sigma}} = \mathcal{O}(\omega^{-1}) \cdot \omega = \mathcal{O}(1), \ k = 0$$

$$\|R^{12}_{\rm KG}(\omega)\|_{H^0_\sigma \to H^1_{-\sigma}} = \mathcal{O}(1), \ k = 1$$

$$\|R^{21}_{\rm KG}(\omega)\|_{H^1_\sigma \to H^0_{-\sigma}} = \mathcal{O}(\omega^{-2}) \cdot \omega^2 = \mathcal{O}(1), \ k = -1$$

Hence,

$$\|R_{\rm KG}(\omega)\|_{E_\sigma \to E_{-\sigma}} = \mathcal{O}(1), \quad \omega \to \infty, \ \ \omega \in \mathbb{C} \setminus [0, \infty)$$

i.e. we have no decay for this resolvent.

We have no decay for the derivatives of the free resolvent too. Moreover, the perturbed resolvent and its derivatives has the same asymptotics as free resolvent. Then we can't integrate by parts. But we have obtained the desired decay for the solution to the free KGE using the explicit formula. Namely,

$$\|W(t)\Psi(0)\|_{E_{-\sigma}} \le C(1+|t|)^{-3/2}\|\Psi(0)\|_{E_\sigma}, \quad \sigma > 3/2$$

To obtain the high energy decay for the perturbed equation, we apply

1) the Huygens principle
2) the Born series
3) the convolution representation.

Now we derive an useful representation for the perturbed resolvent $\mathcal{R}(\omega) := (\mathcal{H}-\omega)^{-1}$ from *the Born decomposition formula*

$$\mathcal{H}-\omega = \mathcal{H}_0-\omega+\mathcal{V} = (\mathcal{H}_0-\omega)[1+\mathcal{R}_0(\omega)\mathcal{V}]$$

Here $\mathcal{H}_0$ is the free KG operator, $\mathcal{R}_0(\omega) = (\mathcal{H}_0-\omega)^{-1}$ is the free resolvent, and

$$\mathcal{V} = \begin{pmatrix} 0 & 0 \\ -iV & 0 \end{pmatrix}$$

Therefore,

$$\mathcal{R}(\omega) = (\mathcal{H} - \omega)^{-1} = [1 + \mathcal{R}_0(\omega)\mathcal{V}]^{-1}\mathcal{R}_0(\omega)$$

and

$$[1 + \mathcal{R}_0(\omega)\mathcal{V}]\mathcal{R}(\omega) = \mathcal{R}_0(\omega).$$

Hence,

$$\mathcal{R}(\omega) = \mathcal{R}_0(\omega) - \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}(\omega) = \mathcal{R}_0(\omega) - \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega) + \mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}(\omega)$$

Taking the inverse Fourier-Laplace transform

$$\Psi(t) = \frac{1}{2\pi i} \int e^{i\omega t}\mathcal{R}(\omega + i0)\Psi(0)$$

we obtain

$$\begin{aligned} \Psi(t) &= W(t)\Psi(0) + i\int_0^t W(t-s)\mathcal{V}W(s)\Psi(0)ds \\ &- iF^{-1}_{\omega\to t}\Big[\mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}\mathcal{R}(\omega)\Psi(0)\Big] = \Psi_1(t) + \Psi_2(t) + \Psi_3(t) \end{aligned}$$

I. We already obtain the time decay for the first term $\Psi_1(t) = W(t)\Psi(0)$.

II. For the second term $\Psi_2(t)$ the decay follows by estimates for the convolution. We have for $\sigma > 3/2$, $\beta > 3$ and $3/2 < \sigma_1 < \min\{\sigma, \beta/2\}$

$$\|W(t-s)\mathcal{V}W(s)\Psi(0)\|_{E_{-\sigma}} \leq \|W(t-s)\mathcal{V}W(s)\Psi(0)\|_{E_{-\sigma_1}}$$

$$\leq \frac{C\|\mathcal{V}W(s)\Psi(0)\|_{E_{\sigma_1}}}{(1+|t-s|)^{3/2}} \leq \frac{C\|W(s)\Psi(0)\|_{E_{-\sigma_1}}}{(1+|t-s|)^{3/2}}$$

$$\leq \frac{C\|\Psi(0)\|_{E_{\sigma_1}}}{(1+|t-s|)^{3/2}(1+|s|)^{3/2}} \leq \frac{C\|\Psi(0)\|_{E_{\sigma}}}{(1+|t-s|)^{3/2}(1+|s|)^{3/2}}$$

Therefore, integrating here in s, we obtain that

$$\|\Psi_2(t)\|_{E_{-\sigma}} \leq \frac{C\|\Psi(0)\|_{E_{\sigma}}}{(1+|t|)^{3/2}}, \quad t \in \mathbb{R}, \quad \sigma > 3/2.$$

III. Consider the last term $\Psi_3(t)$. We have

$$\begin{aligned}\Psi_3(t) &= \frac{1}{2\pi i}\int\limits_\Gamma e^{-i\omega t}\chi(\omega)\Big[\mathcal{R}_0\mathcal{V}\mathcal{R}_0\mathcal{V}\mathcal{R}(\omega+i0)-\mathcal{R}_0\mathcal{V}\mathcal{R}_0\mathcal{V}\mathcal{R}(\omega-i0)\Big]\Psi_0\,d\omega \\ &+ \frac{1}{2\pi i}\int\limits_\Gamma e^{-i\omega t}(1-\chi(\omega))\Big[\mathcal{R}_0\mathcal{V}\mathcal{R}_0\mathcal{V}\mathcal{R}(\omega+i0)-\mathcal{R}_0\mathcal{V}\mathcal{R}_0\mathcal{V}\mathcal{R}(\omega-i0)\Big]\Psi_0\,d\omega \\ &= I_1(t)+I_2(t)\end{aligned}$$

where $\chi(\omega)\in C_0^\infty(\mathbb{R})$, $\chi(\omega)=1$ in the vicinity of $\pm m$.
The matrix $\mathcal{V}\mathcal{R}_0(\omega)\mathcal{V}=\mathcal{V}R_{\rm KG}(\omega)\mathcal{V}$ has the lucky structure:

$$\mathcal{V}R_{\rm KG}(\omega)\mathcal{V}=\begin{pmatrix} 0 & 0\\ -iVR_{\rm S}(\omega^2-m^2)V & 0\end{pmatrix}$$

and the theorem of Agmon with $k=-1$ implies

$$\|R_{\rm S}(\omega)\|_{H^1_\sigma\to H^0_{-\sigma}}=\mathcal{O}(\omega^{-1}),\ \omega\to\infty,\qquad \omega\in\mathbb{C}\setminus[0,\infty),\quad \sigma>1/2$$

Then

$$\|R_{\rm S}(\omega^2-m^2)\|_{H^1_\sigma\to H^0_{-\sigma}}=\mathcal{O}(|\omega|^{-2}),\ \omega\to\infty,\quad \omega\in\mathbb{C}\setminus[0,\infty)$$

We obtain the same decay for the first and second derivatives with $\sigma > 3/2$ and $\sigma > 5/2$, respectively, since

$$\frac{\partial}{\partial\omega} R_{\rm S}(\omega^2 - m^2) = 2\omega R'_{\rm S}(\omega^2 - m^2)$$

Hence, two times partial integration implies that

$$\|I_2(t)\|_{E_{-\sigma}} \leq \frac{C\|\Psi_0\|_{E_\sigma}}{(1+|t|)^2}, \quad t \in \mathbb{R}, \quad \sigma > 5/2.$$

The time decay for $I_1(t)$ follows by the Jensen-Kato technique. Denote

$$\Phi(\omega) = \chi(\omega)\Big[\mathcal{R}_0\mathcal{V}\mathcal{R}_0\mathcal{V}\mathcal{R}(\omega + i0) - \mathcal{R}_0\mathcal{V}\mathcal{R}_0\mathcal{V}\mathcal{R}(\omega - i0)\Big]$$

The following asymptotics hold as $\omega \to \pm m$, $\omega \in \Gamma$:

$$\begin{array}{rcl}
\|\Phi(\omega)\|_{E_\sigma \to E_{-\sigma}} &=& \mathcal{O}(|\omega \mp m|^{1/2}) \\
\|\Phi'(\omega)\|_{E_\sigma \to E_{-\sigma}} &=& \mathcal{O}(|\omega \mp m|^{-1/2}) \\
\|\Phi''(\omega)\|_{E_\sigma \to E_{-\sigma}} &=& \mathcal{O}(|\omega \mp m|^{-3/2})
\end{array}$$

for any $\sigma > 5/2$.

Then we obtain that

$$\|I_1(t)\|_{E_\sigma\to E_{-\sigma}} \le C(1+|t|)^{3/2}, \quad t\in\mathbb{R}$$

Finally

$$\|\Psi_3(t)\|_{E_\sigma\to E_{-\sigma}} \le C(1+|t|)^{3/2}, \quad t\in\mathbb{R}$$

The 1D case is more difficult, since the solutions of free Klein-Gordon equation have slow decay in weighed energy norm. In this case

$$W(t,x-y,) = \left(\begin{array}{cc} \dot U(t,x-y,) & U(t,x-y, \\ \ddot U(t,x-y) & \dot U(t,x-y) \end{array}\right)$$

$$U(t,z) = \frac{1}{2}\theta(t-|z|)J_0(m\sqrt{t^2-z^2}) \sim t^{-1/2}, \;\; |z|\le \varepsilon t, \;\; 0<\varepsilon<1$$

Hence,

$$\|W(t)\|_{E_\sigma\to E_{-\sigma}} = \mathcal{O}(t^{-1/2}), \;\; t\to\infty$$

and we can't apply the Born series and the convolution. This difficulty is well known, and it hindered the study of asymptotic stability of many important 1D problems. The slow decay is caused by "zero resonance function" $\psi(x)$=const corresponding to the edge point $\lambda = 0$ of the continuous spectrum of the 1D Schrödinger operator $-d^2/dx^2$. Equivalently, the resolvent of 1D Schrödinger operator is unbounded at $\omega = 0$:

$$R_{\rm S}(\omega, x-y) = \frac{e^{i\sqrt{\omega}|x-y|}}{2i\sqrt{\omega}}$$

The main idea of our approach is spectral analysis of the "bad" term, with the slow decay $\sim t^{-1/2}$. We show that the bad term does not contribute to the high energy component. More precisely, if $\Psi_0 \in E_\sigma$ then the high energy component of solution to the free KGE

$$\Psi_h(t) = \frac{1}{2\pi i}\int\limits_\Gamma \zeta(\omega)e^{-i\omega t}[R_{KG}(\omega + i0) - R_{KG}(\omega - i0)]\Psi(0)d\omega$$

where $\zeta(\omega) = 0$ for $|\omega| \le m+1$, decays $\sim t^{-3/2}$ in $E_{-\sigma}$. Then the decay for high energy component of solution to the perturbed equation follows by our 3D approach. On the other hand, the decay $\sim t^{-3/2}$ for the low energy component in the non-resonant case we can obtain by method Jensen-Kato (see [KK1]).

[KK1] A. Komech, E. Kopylova, Weighted energy decay for 1D Klein-Gordon equation, *Comm. PDE* **35** (2010), no.2, 353-374.