1 1D Klein-Gordon equation. Weighted energy decay

$$\left\{\begin{array}{l} \ddot\psi(x,t) = -(H+m^2)\psi(x,t), \quad (x,t)\in\mathbb{R}^2, \quad H = -\frac{\partial^2}{\partial x^2} + V(x) \\ \\ \psi(x,0)=\psi_0(x), \quad \dot\psi(x,0)=\pi_0(x) \end{array}\right.$$

$$i\dot\Psi(t) = \mathcal{H}\Psi(t)$$

$$\mathcal{H} = i\begin{pmatrix} 0 & 1 \\ -(H+m^2) & 0 \end{pmatrix}, \quad \Psi(t) = \begin{pmatrix} \psi(t) \\ \dot\psi(t) \end{pmatrix}, \quad \Psi(0) = \begin{pmatrix} \psi_0 \\ \pi_0 \end{pmatrix}$$

Denote

$$\mathcal{M}_t(k) = \begin{pmatrix} \cos(t\sqrt{k^2+m^2}) & \dfrac{\sin(t\sqrt{k^2+m^2})}{\sqrt{k^2+m^2}} \\ \\ -\sqrt{k^2+m^2}\sin(t\sqrt{k^2+m^2}) & \cos(t\sqrt{k^2+m^2}) \end{pmatrix}$$

$$[M_t(k)]^{12} \sim k^{-1}, \quad [M_t(k)]^{11}, [M_t(k)]^{22} \sim 1, \; [M_t(k)]^{21} \sim k, \; k\to\infty$$

Then

$$[e^{-it\mathcal{H}}\mathcal{P}_c](x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{M}_t(k)\,e^{i|y-x|k}(\psi(x,y,k)+1)dk$$

In particular, for $\Psi(0) \in X_c(\mathcal{H})$, we obtain

$$\|\psi(t)\|_{L^\infty} \le Ct^{-1/2}\Big(\|\psi_0\|_{H^{\frac{3}{2},1}} + \|\pi_0\|_{H^{\frac{1}{2},1}}\Big)$$

The energy conservation

$$\int (|\psi'(t)|^2 + m^2|\psi(t)|^2 + |\dot\psi(t)|^2)dx = \text{Const}$$

implies that $\|\Psi(t)\|_{H^1\oplus L^2} \le C\|\Psi(0)\|_{H^1\oplus L^2}$

So it is natural to study dispersion decay in **weighted energy norms**.

Definition $H^1_\sigma = H^1_\sigma(\mathbb{R})$, $\sigma \in \mathbb{R}$

$$\|\psi\|_{H^1_\sigma} = \|\langle x\rangle^\sigma\psi\|_{L^2(\mathbb{R})} + \|\langle x\rangle^\sigma\psi'\|_{L^2(\mathbb{R})} < \infty, \quad \langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$$

$$E_\sigma := H^1_\sigma \oplus L^2_\sigma$$

First we consider $V = 0$.

$$i\dot{\Psi}(t) = \mathcal{H}\Psi(t), \quad \Psi(t) = \begin{pmatrix} \psi(t) \\ \dot{\psi}(t) \end{pmatrix}, \quad \mathcal{H} = i \begin{pmatrix} 0 & 1 \\ \Delta - m^2 & 0 \end{pmatrix}$$

$W(t)$ is the dynamical group of free KGE:

$$\Psi(t) = W(t)\Psi(0)$$

$$W(t,x,y) = W(t,x-y) = \begin{pmatrix} \dot{U}(t,x-y) & U(t,x-y) \\ \ddot{U}(t,x-y) & \dot{U}(t,x-y) \end{pmatrix}, \quad x,y \in \mathbb{R}$$

$$U(t,z) = \frac{1}{2}\theta(t-|z|)J_0(m\sqrt{t^2-z^2})$$

For any $\varepsilon \in (0,1)$

$$|W^{ij}(t,z)| + |\frac{\partial}{\partial z}W^{ij}(t,z)| \leq C(\varepsilon)(1+|t|)^{-1/2}, \quad |z| \leq \varepsilon|t| \tag{1.1}$$

Theorem $\|W(t)\|_{E_{-\sigma}\to E_\sigma} = \mathcal{O}(t^{-1/2}), \quad t\to\infty, \quad \sigma > 1/2$ ([KK10])

Proof We fix $0<\varepsilon<1$, and split $\Psi_0 = \Psi'_{0,t} + \Psi''_{0,t}$ such that

$$\|\Psi'_{0,t}\|_{E_\sigma} + \|\Psi''_{0,t}\|_{E_\sigma} \le C\|\Psi_0\|_{E_\sigma}, \quad t\ge 1 \tag{1.2}$$

and

$$\begin{cases} \Psi'_{0,t}(x) = 0 \quad \text{for} \quad |x| > \frac{\varepsilon t}{2} \\ \Psi''_{0,t}(x) = 0 \quad \text{for} \quad |x| < \frac{\varepsilon t}{4} \end{cases}$$

The energy conservation and (1.2) imply for $t\ge 1$ and $\sigma > 1/2$

$$\begin{aligned} 1)\ \|W(t)\Psi''_{0,t}\|_{E_{-\sigma}} &\le \|W(t)\Psi''_{0,t}\|_{E_0} \le C\|\Psi''_{0,t}\|_{E_0} \\ &\le C_1 t^{-\sigma}\|\Psi''_{0,t}\|_{E_\sigma} \le C_2 t^{-1/2}\|\Psi_0\|_{E_\sigma}, \quad C_j = C_j(\varepsilon) \end{aligned}$$

$$\|\pi''_{0,t}\|^2_{L^2} = \int |\pi''_{0,t}|^2 dx = \int_{|x|\ge\frac{\varepsilon t}{4}} \langle x\rangle^{-2\sigma}\langle x\rangle^{2\sigma}|\pi''_{0,t}|^2 dx \le C\langle\varepsilon t\rangle^{-2\sigma}\|\pi''_{0,t}\|^2_{L^2_\sigma}$$

$$\text{Let}\quad \zeta(s)\in C_0^\infty(\mathbb{R}):\quad \zeta(s)=\begin{cases}1, & |s|<\varepsilon/4\\ 0, & |s|>\varepsilon/2\end{cases}$$

and let ζ be the operator of multiplication by the function $\zeta(|x|/t)$. Then

$$\begin{aligned}2)||(1-\zeta)W(t)\Psi'_{0,t}||_{E_{-\sigma}} \le\ & Ct^{-\sigma}||(1-\zeta)W(t)\Psi'_{0,t}||_{E_0} \le C_1t^{-\sigma}||W(t)\Psi'_{0,t}||_{E_0}\\ \le\ & C_2t^{-\sigma}||\Psi'_{0,t}||_{E_0} \le C_3t^{-1/2}||\Psi_0||_{E_\sigma}\end{aligned}$$

since $1-\zeta(|x|/t)=0$ for $|x|<\varepsilon t/4$

Let $\chi_{\varepsilon t/2}$ be the characteristic function of $[-\varepsilon t/2, \varepsilon t/2]$. Then

$$\zeta W(t)\Psi'_{0,t} = \zeta W(t)\chi_{\varepsilon t/2}\Psi'_{0,t}$$

$\tilde{W}(t, x-y) = \zeta(|x|/t)W(t, x-y)\chi_{\varepsilon t/2}(y)$ is the matrix kernel of this operator. Note that $\tilde{W}(t, x-y) = 0$ for $|x| > \varepsilon t/2$ and for $|y| > \varepsilon t/2$, then (1.1) implies

$$\sup_{x,y} |\partial_x^j \tilde{W}(t, x-y)| \le Ct^{-1/2}, \quad j = 0, 1, \quad t \ge 1$$

The norm of $\zeta W(t)\chi_{\varepsilon t/2} : E_\sigma \to E_{-\sigma}$ is equivalent to the norm of

$$\langle x\rangle^{-\sigma}\zeta W(t)\chi_{\varepsilon t/2}(y)\langle y\rangle^{-\sigma} : E_0 \to E_0$$

It is the Hilbert-Schmidt operator for $\sigma > 1/2$, and

3) $\quad ||\zeta W(t)\Psi'_{0,t}||_{E_{-\sigma}} \le Ct^{-1/2}||\Psi'_{0,t}||_{E_\sigma} \le Ct^{-1/2}||\Psi_0||_{E_\sigma}, \quad t \ge 1$

[KK10] A. Komech, E. Kopylova, Weighted energy decay for 1D Klein-Gordon equation, Comm. PDE 35 (2010), no.2, 353-374.

$V \neq 0$

Theorem 1.1 *(resonant case) Let* $|V(x)| + |V'(x)| \leq C(1+|x|)^{-\beta}$, *with* $\beta > 2$ *and let* $\Psi(0) \in E_\sigma$, $\sigma > 3/2$. *Then*

$$\|\mathcal{P}_c \Psi(t)\|_{E_{-\sigma}} = \mathcal{O}(|t|^{-1/2}), \quad t \to \pm\infty.$$

Theorem 1.2 *(non-resonant case) Let* $|V(x)| + |V'(x)| \leq C(1+|x|)^{-\beta}$, *with* $\beta > 3$ *and let* $\Psi(0) \in E_\sigma$, $\sigma > 5/2$. *Then*

$$\|\mathcal{P}_c \Psi(t)\|_{E_{-\sigma}} = \mathcal{O}(|t|^{-3/2}), \quad t \to \pm\infty.$$

[KK10] A. Komech, E. Kopylova, Weighted energy decay for 1D Klein-Gordon equation, Comm. PDE 35 (2010), no.2, 353-374.

2 3D Schrödinger equation

Theorem (Theorem 2, [GS]) $|V(x)| \le C(1+|x|)^{-\beta}$ with $\beta > 3$, point 0 is neither an eigenvalue nor a resonance of $H = -\Delta + V$. Then

$$\|e^{-itH}\|_{L^1 \to L^\infty} = \mathcal{O}(t^{-3/2}), \quad t \to \infty \tag{2.3}$$

In the case $V = 0$, $e^{-itH}(x,y) = \frac{1}{(4\pi it)^{3/2}} e^{-\frac{|x-y|^2}{4it}}$, and (2.3) holds.

Corollary $\|e^{-itH}\|_{L^2_\sigma \to L^2_{-\sigma}} = \mathcal{O}(t^{-3/2}), \quad t \to \infty, \ \sigma > 3/2$

$$\Big(\int_{\mathbb{R}^2} |\langle x\rangle^{-\sigma} e^{-itH}(x,y)\langle y\rangle^{-\sigma}|^2 dxdy\Big)^{\frac12} \le \sup_{x,y} |e^{-itH}(x,y)| \Big(\int_{\mathbb{R}^2} \langle x\rangle^{-2\sigma}\langle y\rangle^{-2\sigma} dxdy\Big)^{\frac12}$$

This decay was obtained for the first time in [JK1979] for $\sigma > 5/2$.

[GS] M.Goldberg, W.Schlag, Dispersive estimates for Schrödinger operators in dimensions one and three, *Comm.Math.Phys.* **251** (2004), 157-178
[JK1979] A.Jensen, T.Kato, Spectral properties of Schrödinger operators and time-decay of the wave functions, *Duke Math.J.* **46** (1979), 583-611

3 3D Klein-Gordon equation

$$i\dot{\Psi}(t) = \mathcal{H}\Psi(t), \quad \mathcal{H} = i\begin{pmatrix} 0 & 1 \\ \Delta - m^2 - V & 0 \end{pmatrix}$$

Definition $E_\sigma := H^1_\sigma \oplus L^2_\sigma, \quad H^s_\sigma = H^s_\sigma(\mathbb{R}^3),\ s, \sigma \in \mathbb{R}$

$$\|\psi\|_{H^s_\sigma} = \|\langle x\rangle^\sigma \langle\nabla\rangle^s \psi\|_{L^2(\mathbb{R}^3)} < \infty, \quad \langle x\rangle = (1 + |x|^2)^{\frac{1}{2}}$$

V=0. Similar to 1D case, we deduce the decay from explicit formulas for Green function. Let $W(t)$ be the dynamical group of the free Klein-Gordon equation: $\Psi(t) = W(t)\Psi(0)$

$$\text{where} \quad W(t,x,y) = W(t,x-y) = \begin{pmatrix} \dot{U}(t,x-y) & U(t,x-y) \\ \ddot{U}(t,x-y) & \dot{U}(t,x-y) \end{pmatrix}$$

$$U(t,z) = \frac{\delta(|t|-|z|)}{4\pi t} - \frac{m}{4\pi}\frac{\theta(|t|-|z|)J_1(m\sqrt{t^2-|z|^2})}{\sqrt{t^2-|z|^2}}$$

$$|W(t,z)| \le C(\varepsilon)(1+|t|)^{-3/2}, \quad |z| \le \varepsilon|t|, \quad \varepsilon \in (0,1)$$

Hence for $\sigma > 3/2$

$$\|W(t)X(0)\|_{E_{-\sigma}} \le C(1+|t|)^{-3/2}\|X(0)\|_{E_\sigma}$$

$V \neq 0$. We assume that

$$(\mathbf{1}) \quad |V(x)| + |\nabla V(x)| \le C(1+|x|)^{-\beta}, \quad \beta > 3, \quad x \in \mathbb{R}^3$$

$(\mathbf{2}) \quad \lambda = 0 \quad$ is neither eigenvalue nor resonance of $H = -\Delta + V(x)$

Theorem 3.1 *([KK]) Let* $\Psi(0) \in E_\sigma$, $\sigma > 5/2$. *Then*

$$\|\mathcal{P}_c\Psi(t)\|_{E_{-\sigma}} = \mathcal{O}(|t|^{-3/2}), \quad t \to \pm\infty.$$

[KK] A. Komech, E. Kopylova, Weighted energy decay for 3D Klein-Gordon equation, *J. Diff. Eqns.* **248** (2010), no. 3, 501-520.

A. Jensen and T. Kato approach. Free Schrödinger equations ($V \equiv 0$).

$$i\dot{\psi}(x,t) = -\Delta\psi(x,t), \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}$$

$$R_S(\omega) = (-\Delta - \omega)^{-1}, \quad R_S(\omega, x-y) = \frac{e^{i\sqrt{\omega}|x-y|}}{4\pi|x-y|}, \quad \omega \in \mathbb{C} \setminus [0,\infty)$$

$$\psi(t) = \frac{1}{2\pi i} \int\limits_{[0,\infty)} e^{-i\omega t}[R_S(\omega + i0) - R_S(\omega - i0)]\psi(0)d\omega$$

$$= \frac{1}{\pi i} \int\limits_{[0,\infty)} e^{-i\omega t}\,[\mathrm{Im} R_S(\omega + i0)]\psi(0)d\omega$$

since $R_S(\omega - i0) = \overline{R_S(\omega + i0)}$.

Low frequencies

$$R_S(\omega)=\frac{e^{i\sqrt{\omega}|x-y|}}{4\pi|x-y|},\quad R_S'(\omega)=\frac{ie^{i\sqrt{\omega}|x-y|}}{8\sqrt{\omega}},\quad R_S''(\omega)=-\frac{|x-y|e^{i\sqrt{\omega}|x-y|}}{16\omega}-\frac{e^{i\sqrt{\omega}|x-y|}}{16\omega\sqrt{\omega}}$$

$$R_S(\omega\pm i0)=\mathcal{O}(1),\quad \omega\to 0 \text{ in } L^2_\sigma\to L^2_{-\sigma},\quad \sigma>3/2$$
$$R_S'(\omega\pm i0)=\mathcal{O}(\omega^{-1/2}),\quad \omega\to 0 \text{ in } L^2_\sigma\to L^2_{-\sigma},\quad \sigma>3/2$$
$$R_S''(\omega\pm i0)=\mathcal{O}(\omega^{-3/2}),\quad \omega\to 0 \text{ in } L^2_\sigma\to L^2_{-\sigma},\quad \sigma>5/2$$

The last asymptotics follows from:

$$\int\langle x\rangle^{-2\sigma}(1+|x|)^2(1+|y|)^2\langle y\rangle^{-2\sigma}\le C,\ \text{if } 2\sigma-2>3$$

since

$$(|x-y|\le(1+|x|)(1+|y|))$$

Contribution of law frequencies decays $\sim t^{-3/2}$ in $L^2_\sigma\to L^2_{-\sigma}$ with $\sigma>5/2$.

Let $\mathbf{B}$ denote a Banach space with the norm $\|\cdot\|$, and $b > a$.

Lemma 3.2 *(Lemma 10.2 [JK]) Let $F \in C([a,b],\mathbf{B})$ satisfy*

$$F(a) = F(b) = 0, \qquad F'' \in L^1([a+\delta,b],\mathbf{B}), \quad \forall\delta > 0$$

$$\|F''(\omega)\| = \mathcal{O}(|\omega - a|^{-3/2}), \quad \omega \to a$$

Then

$$\int\limits_a^b e^{-it\omega} F(\omega) d\omega = \mathcal{O}(t^{-3/2}), \quad t \to \infty \text{ in } \mathbf{B}$$

Proof Extending F by $F(\omega) = 0$ for $\omega < a$ and for $\omega > b$, we obtain a continuous operator function F on $(-\infty, \infty)$ with $F' \in L^1((-\infty, \infty), \mathbf{B})$.

Due to Zygmund's formula ([Z]), we have

$$\int_{-\infty}^{\infty} F'(\omega)e^{-it\omega}d\omega = -\frac{1}{2}\int_{-\infty}^{\infty}(F'(\omega + \frac{\pi}{t}) - F'(\omega))e^{-it\omega}d\omega$$

Indeed $\int f(\omega + x)e^{-it\omega}d\omega = \int f(u)e^{-itu}e^{itx}du$, and $e^{it\frac{\pi}{t}} = e^{i\pi} = -1$. Furthermore,

$$\int_{-\infty}^{\infty} \|F'(\omega + \frac{\pi}{t}) - F'(\omega)\|d\omega = \int_{-\infty}^{a+\pi/t} + \int_{a+\pi/t}^{\infty} \leq 2\int_{a}^{a+2\pi/t} \|F'(\omega)\|d\omega$$

$$+ \int_{a+\pi/t}^{\infty} d\omega \int_{\omega}^{\omega+\pi/t} \|F''(\nu)\|d\nu = \mathcal{O}(t^{-1/2}) + \frac{\pi}{t}\int_{a+\pi/t}^{\infty} \|F''(\nu)\|d\nu = \mathcal{O}(t^{-1/2})$$

[37] A Zygmund, Trigonometric Series, Cambridge University Press, Cambridge, 1988

We have

$$\int_a^b F'(\omega)e^{-it\omega}d\omega = \mathcal{O}(t^{-1/2})$$

Integrating by part, we get

$$\int_a^b F(\omega)e^{-it\omega}d\omega = \frac{i}{t}\int_a^b F(\omega)de^{-it\omega} = \frac{i}{t}\int_a^b F'(\omega)e^{-it\omega} = \mathcal{O}(t^{-3/2})$$

Free 3D Klein-Gordon equation

$$\Psi(t)=\frac{1}{2\pi i}\int\limits_{\Gamma} e^{-i\omega t}\Big[R_{KG}(\omega+i0)-R_{KG}(\omega-i0)\Big]\Psi(0)d\omega,\quad \Psi=\begin{pmatrix}\psi\\ \dot\psi\end{pmatrix}$$

$$R_{KG}(\omega,x-y)=\frac{e^{i\sqrt{\omega^2-m^2}|x-y|}}{4\pi|x-y|}\begin{pmatrix}\omega & i\\ -i\omega^2 & \omega\end{pmatrix}$$

$$\Gamma=(-\infty,-m]\cup[m,\infty)$$

$$R_{\rm KG}(\omega+i0)-R_{\rm KG}(\omega-i0)\sim\sqrt{\omega\mp m},\ \omega\to\pm m$$

Contribution of low frequencies decays $\sim t^{-3/2}$ as $t\to\infty$.