1 1D Klein-Gordon equation

$$\ddot{\psi}(x,t) = -(H+m^2)\psi(x,t), \quad (x,t)\in\mathbb{R}^2, \quad H = -\frac{\partial^2}{\partial x^2} + V(x)$$
$$\psi(x,0) = 0, \quad \dot{\psi}(x,0) = \pi_0(x)$$

Introduce the operator ("Bessel potential")

$$\mathcal{J}_\alpha = \mathcal{F}^{-1}\langle\cdot\rangle^\alpha\mathcal{F}, \quad \alpha\in\mathbb{R}, \quad \langle x\rangle = (1+|x|^2)^{1/2}$$

where $\mathcal{F}$ is FT. The generalized Sobolev space $H^{\alpha,1}_\sigma = H^{\alpha,1}_\sigma(\mathbb{R})$ (cf. [BL]) is the space of all tempered distributions $f\in\mathcal{S}'(\mathbb{R})$ with finite norm

$$\|f\|_{H^{\alpha,1}_\sigma} = \|\mathcal{J}_\alpha f\|_{L^1_\sigma}, \qquad \alpha,\sigma\in\mathbb{R} \tag{1.1}$$

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As before $H^{\alpha,1} = H^{\alpha,1}_0$.

Remark $H^{k,1} = W^{k,1}$ for $k\in\mathbb{N}$.

[BL] J. Bergh, J.Löfström, Interpolation Spaces, Springer, Berlin, 1976.

Theorem 1.1 *i) Let $V \in L^1_1$, $\pi_0 \in H^{\frac{1}{2},1}$, $\pi_0 \in X_c(H)$. Then*

$$\|\psi(t)\|_{L^\infty} = \mathcal{O}(t^{-1/2}), \quad t \to \infty \tag{1.2}$$

ii) Let $V \in L^1_2$, $\pi_0 \in H_1^{\frac{1}{2},1}$, $\pi_0 \in X_c(H)$. Then, in the non-resonant case

$$\|\psi(t)\|_{L^\infty_{-1}} = \mathcal{O}(t^{-3/2}), \quad t \to \infty \tag{1.3}$$

Here $X_c(H)$ is the continuous spectrum subspace of H.

We note that similar decays is frequently stated in terms of the Besov space $B^{\frac{1}{2}}_{1,1}(\mathbb{R})$. Namely, (1.2) holds with $B^{\frac{1}{2}}_{1,1}$ in place of $H^{\frac{1}{2},1}$ since $B^{\frac{1}{2}}_{1,1} \subset H^{\frac{1}{2},1}$. Similarly, (1.3) holds with $B^{\frac{1}{2}}_{1,1,1}$ in place of $H_1^{\frac{1}{2},1}$, where $B^{\frac{1}{2}}_{1,1,1}$ is the corresponding weighted Besov space.

$W^{k,p} \to L^q$ estimates for the 3D KGEs were established in [SW].

In 1D case $W^{k,p} \to W^{k,q}$ estimates were obtained in [W1] for $V \in L^1_\gamma$, where $\gamma > \frac{3}{2}$ in the non-resonant case and $\gamma > \frac{5}{2}$ in the resonant case. The dispersive estimate of type (1.2) in Besov space is shown in [DF] but again requiring $V \in L^1_2$ in the resonant case.

[SW] A. Soffer, M.I. Weinstein, Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations, *Invent.Math.* **136** (1999), 9-74.

[W1] R. Weder, Inverse scattering on the line for the nonlinear Klein–Gordon equation with a potential, *J. Math. Anal. Appl.* **252** (2000), 102–123.

[AF] P. D'Ancona, L. Fanelli, L^p- boundedness of the wave operator for the one dimensional Schrödinger operator, *Comm. Math. Phys.* **268** (2006), 415–438.

Free Klein–Gordon equation $(V = 0)$

$$\ddot{\psi}(x,t) = \psi''(x,t) - m^2\psi(x,t), \quad \psi(x,0) = 0, \quad \dot{\psi}(x,0) = \pi_0(x),$$

$$\text{FT}: \qquad \hat{\psi}(k,t) = \frac{\sin(t\sqrt{k^2+m^2})}{\sqrt{k^2+m^2}}\hat{\pi}_0(k)$$

$$\psi(x,t) = U(t)\pi_0(x)$$

$$U(t,x,y) = \frac{1}{2\pi}\int e^{ik(x-y)}\frac{\sin(t\sqrt{k^2+m^2})}{\sqrt{k^2+m^2}}dk = \frac{1}{2}\theta(t^2-|x-y|^2)J_0(\sqrt{t^2-|x-y|^2})$$

$$\|\psi(t)\|_{L^\infty} \leq C(1+|t|)^{-1/2}\Big(\|\pi_0\|_{L^1} + \|\pi_0'\|_{L^1}\Big) \quad [RS]$$

[RS] M. Reed, B. Simon, Methods of modern mathematical physics, III.

We will prove that $\|\psi(t)\|_{L^\infty} \leq C(1+|t|)^{-1/2}\|\pi_0\|_{H^{\frac{1}{2},1}}$

First consider the case $V=0$.

Let $\xi(x) \in C_0^\infty$: $\xi(x) = 1$ for $x \leq 1$ and $\xi(x) = 0$ for $x \geq 2$, and $\zeta(x) = 1 - \xi(x)$. We split

$$U(t,x,y) = U_1(t,x,y) + U_2(t,x,y)$$

where

$$U_1(t,x,y) = \frac{1}{2\pi}\int e^{ik(x-y)}\xi(k^2)\frac{\sin(t\sqrt{k^2+m^2})}{\sqrt{k^2+m^2}}dk$$

$$U_2(t,x,y) = \frac{1}{2\pi}\int e^{ik(x-y)}\zeta(k^2)\frac{\sin(t\sqrt{k^2+m^2})}{\sqrt{k^2+m^2}}dk$$

Low frequencies Consider the oscillatory integral

$$I(t) = \int_a^b e^{it\phi(k)} f(k)dk, \quad \phi(k) \in \mathbb{R}$$

Lemma 3 Let $\phi''(k) \neq 0$ for $k \in [a, b]$ and $f \in \mathcal{A}_1$. Then

$$|I(t)| \leq C[t \min_{a \leq k \leq b} |\phi''(k)|]^{-1/2} \|f\|_{\mathcal{A}_1}, \quad t \geq 1$$

Proof. We have $f(k) = c + \int_{\mathbb{R}} e^{iky} \hat{g}(y)dy, \quad \hat{g}(y) \in L^1$

$$I(t) = \int_{\mathbb{R}} \hat{g}(y) I_{y/t}(t)dy + cI_0(t), \quad I_v(t) = \int_a^b e^{it(\phi(k)+vk)}dk$$

By the van der Corput lemma

$$|I_v(t)| \leq C[t \min_{a \leq k \leq b} |\phi''(k)|]^{-1/2}, \quad t \geq 1, \quad v \in \mathbb{R}$$

and the claim follows from the definition of the norm in $\mathcal{A}_1$:

$$\|f\|_{\mathcal{A}_1} = c + \|\hat{g}\|_{L^1}$$

$$U_1(t,x,y) = \frac{1}{2\pi}\int e^{ik(x-y)}\xi(k^2)\frac{\sin(t\sqrt{k^2+m^2})}{\sqrt{k^2+m^2}}dk$$

is the oscillatory integrals with the phase functions

$$\phi_\pm(k) = \pm\sqrt{k^2+m^2} - vk, \quad v = \frac{y-x}{t}.$$

The second derivative of $\phi_\pm(k)$ satisfies

$$|\phi''_\pm(k)| = \frac{m^2}{\sqrt{(k^2+m^2)^3}} \geq C\frac{m^2}{\sqrt{(2+m^2)^3}} = C(m),$$

Further, $(k^2+m^2)^{-1/2}\xi(k^2) \in C_0^\infty \subset \mathcal{A}$, then Lemma 3 implies

$$\|U_1(t)\|_{L^1\to L^\infty} = \max_{x,y\in\mathbb{R}}|U_1(t,x,y)| \leq Ct^{-1/2}, \quad t \geq 1.$$

High frequencies

We following lemma is the adapted version of Lemma 2 from [MSW]. For the proof we apply Lemma 6.7 from [C].

Lemma 4 Let $\eta(k) \in C^\infty[1,\infty)$, $|\eta^{(j)}(k)| \leq k^{-j}$ for $j = 0, 1$. Then

$$\sup_{p\in\mathbb{R}} \left| \int_1^\infty \eta(k) \frac{e^{\pm it\sqrt{k^2+m^2}+ikp}}{k^{3/2}} dk \right| \leq Ct^{-1/2}$$

[MSW] B. Marshall, W. Strauss, and S. Wainger, *$L^p - L^q$ estimates for the Klein–Gordon equation*, J. Math. Pures et Appl. **59** (1980), 417–440.

[C] S. Cuccagna, On dispersion for Klein Gordon equation with periodic potential in 1D, *Hokkaido Math. J.* **37** (2008), 627–645.

$$\int U_2(t,x,y)f(y)dy = \frac{1}{2\pi}\int\Big(\int \zeta(k^2)\frac{\sin(t\sqrt{k^2+m^2})}{\sqrt{k^2+m^2}}e^{ik(x-y)}dk\Big)f(y)dy$$

$$= \int \zeta(k^2)\frac{\sin(t\sqrt{k^2+m^2})}{\sqrt{k^2+m^2}(1+k^2)^{1/4}}e^{ikx}(1+k^2)^{1/4}\hat{f}(k)dk$$

$$= \frac{1}{4\pi i}\sum_{\mp}\int_{\mathbb{R}} g(y)\Big(\int_1^\infty \eta(k)\frac{e^{\pm it\sqrt{k^2+m^2}+ik(x-y)}}{k^{3/2}}dk\Big)dy$$

$$\text{Here}\quad \eta(k) = \zeta(k^2)\frac{k^{3/2}}{(1+k^2)^{1/4}(k^2+m^2)^{1/2}}, \qquad g = \mathcal{J}_{\frac{1}{2}}f,$$

$$\|U_2(t)f\|_{L^\infty} \le C\sup_{x,y\in\mathbb{R}}\Big|\int_1^\infty \eta(k)\frac{e^{\pm it\sqrt{k^2+m^2}+ik(x-y)}}{k^{3/2}}dk\Big|\int_{\mathbb{R}}|g(y)|dy$$

$$\le Ct^{-1/2}\|g\|_{L^1} = Ct^{-1/2}\|f\|_{H^{\frac{1}{2},1}}$$

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by Lemma 4. Here $\|g\|_{L^1} = \|f\|_{H^{\frac{1}{2},1}}$ by definition (1.1).

The case $V \neq 0$

$$\psi(x,t) = U(t)\pi_0(x)$$

$$U(t,x,y) = \frac{1}{2\pi}\int e^{ik(x-y)}\frac{\sin(t\sqrt{k^2+m^2})}{\sqrt{k^2+m^2}}(1+\psi(x,y,k))dk$$

where

$$\psi(x,y,k) = h_+(y,k)h_-(x,k)T(k) - 1 \in \mathcal{A}$$

The low energy component $U_1(t,x,y)$ estimated in the same way as in the case $V = 0$ (by Lemma 3). To estimate high energy component $U_2(t,x,y)$ we apply the identity $R(\lambda) = R_0(\lambda) - R_0(\lambda)VR(\lambda)$ to obtain

$$U_2(t,x,y) = \frac{1}{2\pi}\int e^{ik(x-y)}\zeta(k^2)\frac{\sin(t\sqrt{k^2+m^2})}{\sqrt{k^2+m^2}}dk$$

$$+\frac{i}{4\pi}\int V(z)\Big(\int_{|k|\geq 1}\zeta(k^2)\frac{\sin(t\sqrt{k^2+m^2})}{\sqrt{k^2+m^2}}\frac{e^{ik(|x-z|+|z-y|)}}{k}(1+\psi(y,z,k))dk\Big)dz.$$

For the last integral we need the following lemma

11 **Lemma 1.2** *Let $\eta(k) \in C^\infty[1,\infty)$, such that $|\eta^{(j)}(k)| \le k^{-j}$ for $j = 0, 1$. Then for any $g(k) \in \mathcal{A}_1$, $\alpha > 3/2$ and $t \ge 1$*

$$\sup_{p\in\mathbb{R}} \left| \int_1^\infty \eta(k) \frac{e^{\pm t\sqrt{k^2+m^2}+ikp}}{k^\alpha} g(k)dk \right| \le C(\alpha)\|g\|_{\mathcal{A}_1} t^{-1/2}. \qquad (1.4)$$ OI-

Proof Consider "+" case. We have the oscillatory integral

$$I_\alpha(t) = \int_1^\infty k^{-\alpha}\eta(k)e^{it\phi(k)}g(k)dk$$

with the phase function $\phi(k) = \sqrt{k^2+m^2} - vk$, $v = -p/t$.

$$I_\alpha(t) = I_\alpha^1(t) + I_\alpha^2(t) = \int_1^t + \int_t^\infty$$

$$|I_\alpha^2(t)| \le \|g\|_{L^\infty} \int_t^\infty k^{-\alpha}dk \le C\|g\|_{\mathcal{A}_1} t^{1-\alpha}.$$

Denote $\Psi(k,t) = \int\limits_1^k e^{it\phi(\tau)} g(\tau) d\tau$. By the van der Corput lemma

$$|\Psi(k,t)|| \le C\|g\|_{\mathcal{A}_1} t^{-1/2} k^{3/2}. \tag{1.5}$$

estpsi

since $\min\limits_{1\le\tau\le k} \phi''(\tau) = \phi''(k) = \frac{m^2}{(\sqrt{k^2+m^2})^3} \ge \frac{C}{k^3}$

Integrating by parts, we get

$$I^1_\alpha(t) = \int_1^t k^{-\alpha}\eta(k) d\Psi(k,t) = \Psi(t,t)\eta(t)t^{-\alpha} - \int_1^t \Psi(k,t) d(\eta(k)k^{-\alpha})$$

$$|I^1_\alpha(t)| \le C|\Psi(t,t)|t^{-\alpha} + \int_1^t |\Psi(k,t)\Lambda(k)|dk, \quad \Lambda(k) = \frac{k\eta'(k) - \alpha\eta(k)}{k^{\alpha+1}}$$

By (1.5) estpsi $|\Psi(k,t)\Lambda(k)| \le C\|g\|_{\mathcal{A}_1} t^{-1/2} k^{3/2} k^{-\alpha-1} = C\|g\|_{\mathcal{A}_1} t^{-1/2} k^{1/2-\alpha}$

$$|I^1_\alpha(t)| \le C\|g\|_{\mathcal{A}_1} \left(t^{1-\alpha} + (1+\alpha)t^{-1/2} \int_1^t k^{1/2-\alpha} dk \right) \le C\|g\|_{\mathcal{A}_1} t^{-1/2}.$$