Asymptotic stability of kinks for nonlinear relativistic Ginzburg-Landau equation

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Abstract

Soliton solutions are important in the study of evolution equations, mainly because they are often easily found numerically, and also because they generally emerge in the long-time asymptotics of solutions of these equations.

Asymptotic stability means that a solution of the equation with initial data close to one of the solitons asymptotically is a sum of a (possibly different) soliton and a dispersive wave solving the corresponding linear equation.

The study of the asymptotic stability of soliton solutions was inspired by the problem of the stability of elementary particles, because the latter may be identified with solitons of nonlinear field equations.

The first results in this direction were obtained by numerical simulation in 1965 by Zabusky and Kruskal for the Korteweg-de Vries (KdV) equation. In 1967 Gardner, Greene, Kruskal, and Miura used the inverse scattering transform to solve the KdV equation analytically. These results were extended to other integrable equations by Its, Khruslov, Shabat, Zakharov, and other.

The asymptotic stability of solitons for a NLS with small initial data and small coefficient of the nonlinear term was proved by Soffer and Weinstein (1990), (1992). Later, Buslaev and Perelman (1995) and Buslaev and Sulem (2003) established this result in the more difficult instance. Now there are a lot of works concerning the asymptotic stability.

We obtain the first result for the relativistic equation.

Model: nonlinear relativistic wave equation

$$\ddot{\psi}(x,t) = \psi''(x,t) + F(\psi(x,t)), \quad x \in \mathbb{R} \tag{1}$$

$$F(\psi) = -U'(\psi)$$

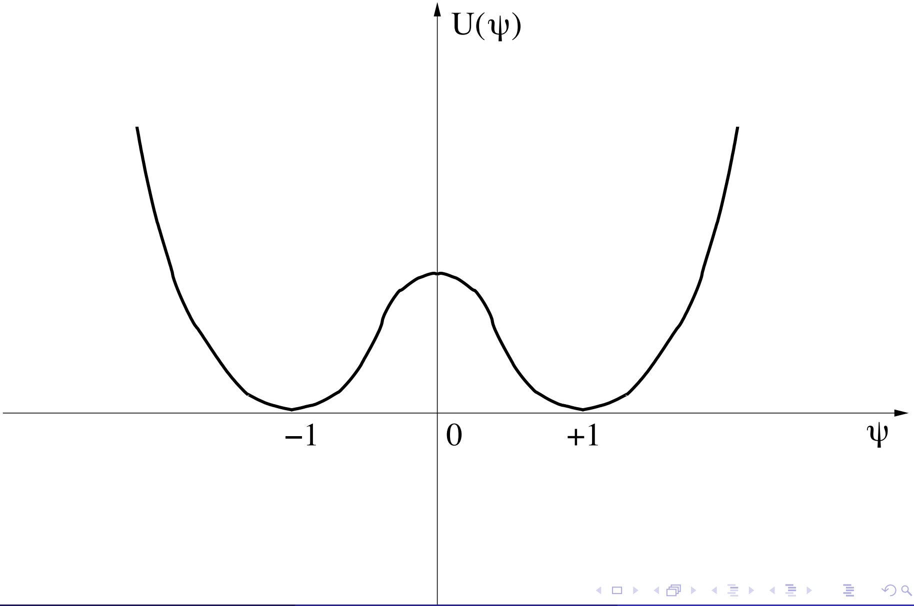
Condition U1 : $\qquad U(\psi) > 0, \quad \psi \neq \pm 1$

$$U(\psi) = \frac{m^2}{2}(\psi \mp 1)^2 + \mathcal{O}(|\psi \mp 1|^{K+2}), \quad \psi \to \pm 1$$

$$K \geq 12$$

The classic Ginzburg-Landau potential: $U(\psi) = \frac{(\psi^2-1)^2}{4}$, $F(\psi) = -\psi^3 + \psi$ does not satisfy **U1** since $K = 3$ then.

Potential of Ginzburg-Landau type



Stationary equation

$$s'' - U'(s) = 0$$

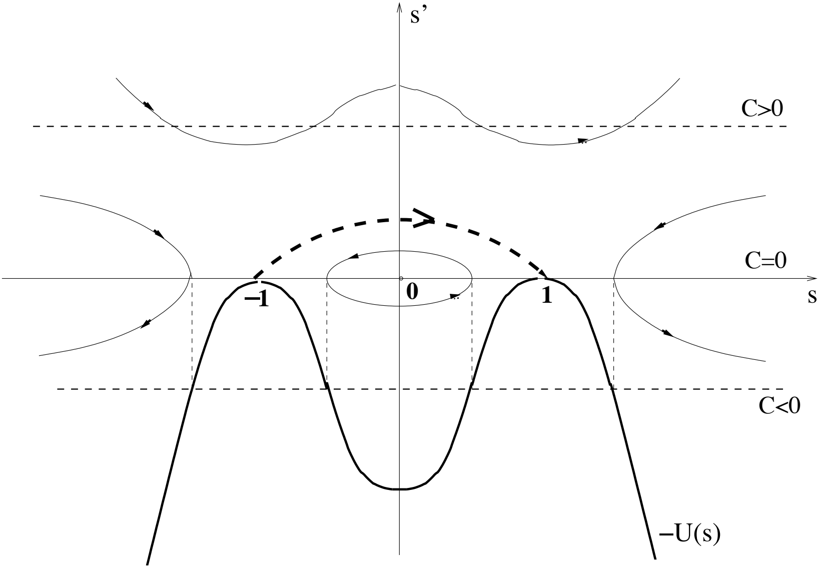
Constant solutions of the stationary equation are

$$s \equiv \pm 1, \quad s \equiv 0$$

Integrating stationary equation, we get

$$\frac{(s')^2}{2} - U(s) = C$$

Phase portrait of stationary equation $\frac{(s')^2}{2} - U(s) = C$



Kink: nonconstant finite energy solution

$$s(x) \to \pm 1 \quad \text{as} \quad x \to \pm\infty$$

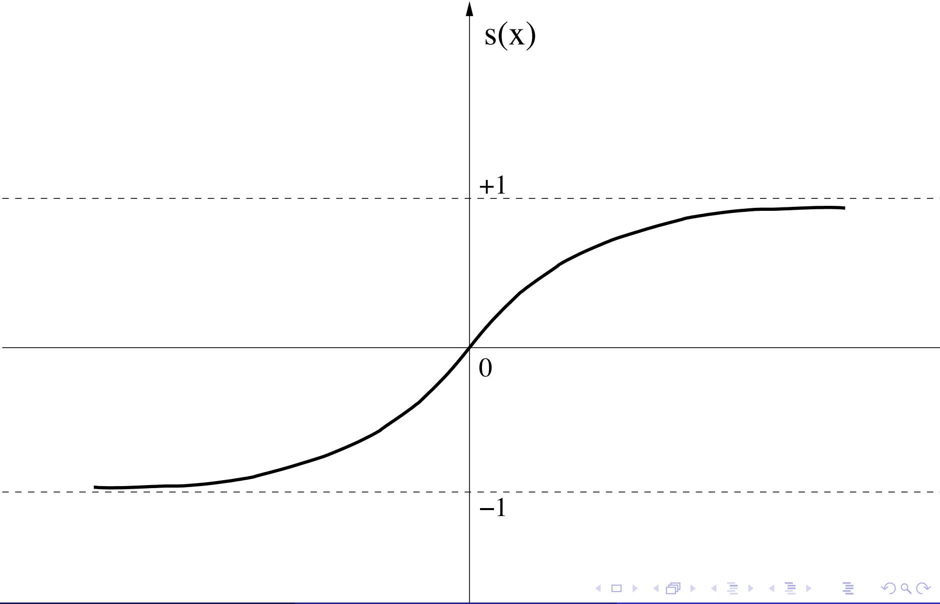
Condition **U1** implies

$$(s(x) \mp 1)'' \sim m^2 (s(x) \mp 1), \quad x \to \pm\infty$$

$$s(x) \mp 1 \sim C e^{-m|x|}, \quad x \to \pm\infty$$

In the case of Ginzburg-Landau potential $\quad s(x) = \tanh \frac{x}{\sqrt{2}}$

Plot of kink



Hamilton equation

In the vector form equation (1) reads

$$\begin{cases} \dot{\psi}(x,t) = \pi(x,t) \\ \dot{\pi}(x,t) = \psi''(x,t) + F(\psi(x,t)), \quad x \in \mathbb{R} \end{cases}$$

$$\dot{Y}(t) = \mathcal{F}(Y(t)), \quad Y(t) = \begin{pmatrix} \psi(t) \\ \pi(t) \end{pmatrix} \tag{2}$$

It is a Hamilton equation, i.e.

$$\dot{Y} = J\mathcal{D}\mathcal{H}(Y), \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\mathcal{H}(Y) = \int \left[\frac{|\pi(x)|^2}{2} + \frac{|\psi'(x)|^2}{2} + U(\psi(x)) \right] dx$$

$\mathcal{D}\mathcal{H}$ is the Fréchet derivative of the Hamilton functional $\mathcal{H}$.

Traveling waves or soliton solutions

$$S_{q,v}(t) = (\psi_v(x - vt - q), \pi_v(x - vt - q))$$

$$q, v \in \mathbb{R}, \qquad |v| < 1$$

$$\psi_v(x) = s(\gamma x), \quad \pi_v = -v\psi_v'(x)$$

$$\gamma = 1/\sqrt{1 - v^2}$$

Definition A solitary manifold is the set $\mathcal{S} := \{S_{q,v} : \; q, v \in \mathbb{R}, \; |v| < 1\}$

Linearization at soliton

$$Y(t) = S_{q,v}(t) + X(t)$$

$$\dot{X}(t) = A_v X(t) + N(t)$$

$$A_v = \begin{pmatrix} v\frac{d}{dx} & 1 \\ \frac{d^2}{dx^2} - m^2 - V_v(x) & v\frac{d}{dx} \end{pmatrix}, \quad V_v(x) = U''(s(\gamma x)) - m^2$$

$$V_v(x) \sim C(s(\gamma x) \mp 1)^K - \sim Ce^{-Km\gamma|x|}, \quad x \to \pm\infty$$

Spectral properties of linearized equation

The determinant of A_v is the Schrödinger operator

$$H_v = -(1 - v^2)\frac{d^2}{dx^2} + m^2 + V_v$$

Spectral properties of H_v are identical for all $v \in (-1, 1)$ by relativictic invariance:

- The continuous spectrum of H_v is $[m^2, \infty)$
- 0 belongs to the discrete spectrum of H_v

$$\varphi_0(x) = s'(\gamma x)$$

Point m^2 is a resonance if there exist a nonzero solution $\psi \in L^\infty(\mathbb{R})$ to

$$H_v \psi = m^2 \psi$$

Condition U2: The edge point m^2 is not resonance of H_v

Condition U3: The discrete spectrum of H_v is $\{0; \mu\}$, where $\frac{m^2}{4} < \mu < m^2$

Condition U4: The Fermi Golden Rule holds

$$\int \varphi_{4\mu}(x) F''(s(x)) \varphi_\mu^2(x) dx \neq 0$$

Main result

Theorem Let conditions **U1** − **U4** hold, and $Y_0 = S_{q_0,v_0} + X_0$, where X_0 is sufficiently small. Then

$$Y(x,t) = (\psi_{v_\pm}(x - v_\pm t - q_\pm), \pi_{v_\pm}(x - v_\pm t - q_\pm))$$

$$+W_0(t)\Phi_\pm + r_\pm(x,t), \quad t \to \pm\infty$$

Here $\Phi_\pm \in E = H^1 \oplus L^2$, and

$$\|r_\pm(t)\|_E = \mathcal{O}(|t|^{-\nu}), \quad t \to \pm\infty, \quad \nu > 0$$

[1] EK, A.Komech, On asymptotic stability of moving kink for relativistic Ginsburg-Landau equation, *Comm. Math. Phys.* (2011)

[2] EK, A.Komech, On asymptotic stability of kink for relativistic Ginsburg-Landau equation, *Arch. Rat. Mech. and Analysis.* (2011)

Classic Ginzburg-Landau potential

$$V_v(x) = \frac{-3}{\cosh^2(\gamma x/\sqrt{2})}$$

- The continuous spectrum is $[2, \infty)$.
- The discrete spectrum is $\{0; 3/2\}$.
- The corresponding Fermi Golden Rule holds

There exists resonance: the function

$$\psi(x) = 1 - 3\tanh^2(\gamma x/\sqrt{2}) \in L^\infty(\mathbb{R})$$

is a solution to $H_v\psi = 2\psi$. Then condition **U2** fails and the asymptotic stability of the kinks in this case is an open problem.

Methods

For the proof we develop the approach of [BS] for NLS

- *Symplectic projection* of the trajectory onto the solitary manifold $\mathcal{S}$
- *Modulation equations* for the parameters of the symplectic projection (Dynamics along $\mathcal{S}$)
- Linearization of the transversal dynamics
- Dispersion decay of linearized dynamics
- Poincaré normal form
- Bounds for majorants.

[BS] V.S.Buslaev, C.Sulem, On asymptotic stability of solitary waves for nonlinear Schrödinger equations, *Ann.Inst.Henri Poincaré, Anal.Non Linéaire* (2003).

Dispersion decay

Klein-Gordon equation

$$\dot\Psi(t) = A_v\Psi(t), \quad \Psi(t) = \begin{pmatrix} \psi(t) \\ \dot\psi(t) \end{pmatrix}, \quad A_v = \begin{pmatrix} v\frac{d}{dx} & 1 \\ -H & v\frac{d}{dx} \end{pmatrix}$$

1. m^2 is not a resonance for the Schrödinger operator
$$H = -\frac{d^2}{dx^2} + m^2 - V(x)$$
2. $|V(x)| \le C(1+|x|)^{-\beta})$ with some $\beta > 3$

Weighted Sobolev spaces $L^p_\sigma = L^p_\sigma(\mathbb{R}), \quad H^1_\sigma = H^1_\sigma(\mathbb{R}), \ \sigma \in \mathbb{R}, \ p = 1, 2, \ldots$

$$\|\psi\|_{L^p_\sigma} = \|\langle x\rangle^\sigma \psi\|_{L^p} < \infty, \quad \|\psi\|_{H^1_\sigma} = \|\langle x\rangle^\sigma \psi\|_{H^1} < \infty, \ \langle x\rangle = (1+|x|^2)^{1/2}$$

Denote $E_\sigma = H^1_\sigma \oplus L^2_\sigma$.

Theorem 1 Let conditions (1) and (2) hold. Then

$$\|e^{A_v t}\mathcal{P}_c\|_{E_\sigma \to E_{-\sigma}} = \mathcal{O}(|t|^{-3/2}), \quad t \to \pm\infty, \ \ \sigma > 5/2$$

Theorem 2 Let $V \in L^1_1$. Then

$$\|[e^{A_v t}\mathcal{P}_c]^{12}\|_{W^{1,1} \to L^\infty} = \mathcal{O}(t^{-1/2}), \quad t \to \infty$$

$[\cdot]^{ij}$ denotes the ij entry of the corresponding matrix operator,
$\|\psi\|_{W^{1,1}} = \|\psi\|_{L^1} + \|\psi'\|_{L^1}$

[1] EK, A.Komech, Weighted energy decay for 1D Klein–Gordon equation, *Comm. PDE* (2010).

[2] I.Egorova, EK, V. A.Marchenko, G.Teschl, Dispersion estimates for one-dimensional Schrödinger and Klein-Gordon equations revisited, *Russian Math. Surveys* (2016).

Symplectic structure

Consider the tangent space $\mathcal{T}_{S_{q,v}}\mathcal{S}$ of the manifold $\mathcal{S}$ at a point $S_{q,v}$. The vectors

$$\begin{aligned} \tau_1 = \tau_1(v) &:= \partial_v S_{q,v} = (-\psi'_v, -\pi'_v) \\ \tau_2 = \tau_2(v) &:= \partial_q S_{q,v} = (\partial_v \psi_v, \partial_v \pi_v) \end{aligned}$$

form a basis in $\mathcal{T}_{S_{q,v}}\mathcal{S}$. The symplectic form Ω

$$\Omega(\tau_1, \tau_2) := \langle \tau_1, J\tau_2 \rangle$$

is nondegenerate on the tangent space $\mathcal{T}_{S_{q,v}}\mathcal{S}$: $\quad \Omega(\tau_1, \tau_2) \neq 0$
i.e. $\mathcal{T}_{S_{q,v}}\mathcal{S}$ is a symplectic subspace.
In a small neighborhood of the soliton manifold $\mathcal{S}$ a “symplectic orthogonal projection” onto $\mathcal{S}$ is well-defined.

Zero eigenspace

$$A_v \tau_1 = 0, \quad A_v \tau_2 = \tau_1$$

Then $Z(t) = c_1\tau_1 + c_2(\tau_1 t + \tau_2)$ is a growing solution to linearized equation

$$\dot{Z}(t) = A_v Z(t)$$

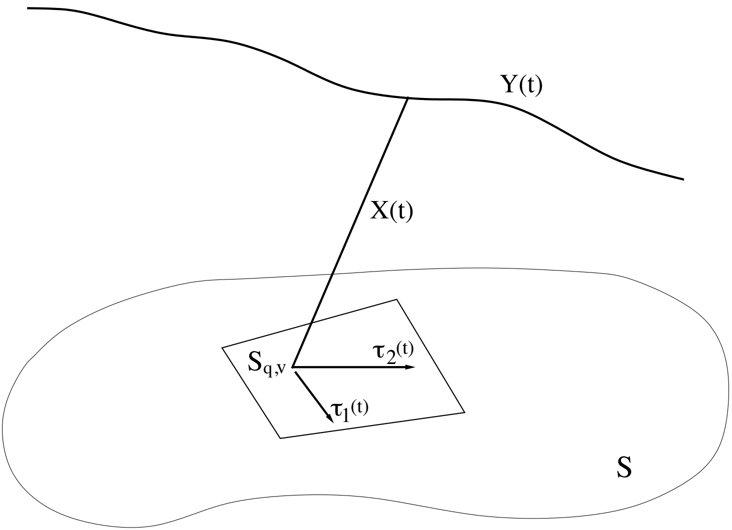
It means the unstable character of dynamics along the solitary manifold. We split a solution to (2) as the sum

$$Y(t) = S_{q(t),v(t)} + X(t),$$

where the symplectic orthogonality condition hold:

$$\Omega(X(t), \tau_1(t)) = \Omega(X(t), \tau_2(t)) = 0$$

Solitary manifold



Transversal dynamics

Differentiating the orthogonality conditions in t we obtain *modulation equations* which implies

$$(\dot{v}(t), \dot{q}(t)) = \mathcal{O}(\|X(t)\|^2) \tag{3}$$

$$Y(t) = S_{q(t),v(t)} + X(t), \quad \dot{X}(t) = A_{v(t)}X(t) + N(t), \quad \|N(t)\| = \mathcal{O}(\|X_1(t)\|^2)$$

$$\dot{X}(t) = A_{v_1}X(t) + Q(t) + N(t), \; v_1 = v(t_1), \quad Q(t) = (A_{v(t)} - A_{v_1})X(t)$$

$$X = (X_1, X_2)$$

Theorem

Let $d_0 := \|X(0)\|_{E_\sigma \cap W} \ll 1, \quad W := W^{2,1} \oplus W^{1,1}$. *Then*

$$\|X(t)\|_{E_{-\sigma}} \le \frac{C(d_0)}{(1+|t|)^{3/2}}, \quad \|X_1(t)\|_{L^\infty} \le \frac{C(d_0)}{(1+|t|)^{1/2}}, \qquad t \ge 0$$

Duhamel representation

$$X(t) = e^{A_1 t} X(0) + \int_0^t e^{A_1(t-s)}[Q(s) + N(s)]ds, \quad A_1 = A_{v_1}$$

$$\|X(t)\|_{E_{-\sigma}} \leq C(1+t)^{-\frac{3}{2}} \|X(0)\|_{E_\sigma} + C \int_0^t (1+t-s)^{-\frac{3}{2}} \|Q(s)+N(s)\|_{E_\sigma} ds \quad (4)$$

$$\|X_1(t)\|_{L^\infty} \leq C(1+t)^{-\frac{1}{2}} \|X(0)\|_W + C \int_0^t (1+t-s)^{-\frac{1}{2}} \|Q(s)+N(s)\|_W ds \quad (5)$$

Equation (3) implies

$$\|Q(t)\|_{E_\sigma \cap W} \leq C \|X(t)\|^2_{E_{-\sigma}}$$

$$N = N_2 + N_3 + ... + N_K + R_{K+1}, \qquad N_j = \frac{F^{(j)}(\psi_v)}{j!} X_1^j$$

$$F(\psi) = -m^2(\psi \mp 1) + \mathcal{O}(|\psi \mp 1|^{K+1}), \quad \psi \to \pm 1$$

Hence $F^{(j)}(\psi_v(x))$, $2 \le j \le K$, decrease exponentially as $|x| \to \infty$, and

$$\| N_j\|_{L^2_\sigma \cap W} \le C\|X_1\|_{L^\infty}\|X_1\|_{L^2_{-\sigma}}, \quad 2 \le j \le K$$

$$\| R_{K+1}\|_{L^2_\sigma} \le C\|X_1\|^K_{L^\infty}\|X_1\|_{L^2_\sigma} \le C(d_0)\|X_1\|^K_{L^\infty} t^{\sigma+3/2}$$

by virial type estimate.

$$\| R_{K+1}\|_W \le C\|X_1\|^{K-1}_{L^\infty}(\|X_1\|^2_{L^2} + \|X_1\|^2_{L^2}\|X_1'\|^2_{L^2} \le C(d_0)\|X_1\|^{K-1}_{L^\infty}$$

we introduce the “majorants”

$$m_1(t) := \sup_{s\in[0,t]} (1+s)^{3/2}\|X(s)\|_{E_{-\sigma}}$$

$$m_2(t) := \sup_{s\in[0,t]} (1+s)^{1/2}\|X_1(s)\|_{L^\infty}$$

$$t_* \text{ is the exit time :} \quad t_* = \sup\{t : m_j(s) < \epsilon, \ \ 0 \le s \le t, \ j = 1, 2\}$$

Note that $m_j(0) < \epsilon$ if $d_0 \ll 1$. Our goal is to prove that $t_* = \infty$ if d_0 is sufficiently small. This would follow if we show that

$$m_j(t) < \epsilon/2, \qquad 0 \le t < t_*, \ \ j = 1, 2.$$

Multiplying (4) by $(1+t)^{3/2}$, and taking the supremum in $t \in [0, t_1]$, we obtain for $t_1 < t_*$

$$m_1(t_1) \le Cd_0 + C \sup_{t\in[0,t_1]} \int_0^t \frac{(1+t)^{\frac{3}{2}} ds}{(1+|t-s|)^{\frac{3}{2}}} \left[\frac{m_1^2(s)}{(1+s)^3} + \frac{m_1(s)m_2(s)}{(1+s)^2} + \frac{m_2^K(s)(1+s)^{\frac{3}{2}+\sigma}}{(1+s)^{K/2}} \right]$$

$m_1(t)$ is a monotone increasing function, then

$$m_1(t_1) \le Cd_0 + C(m_1^2(t_1) + m_1(t_1)m_2(t_1) + m_2^K(t_1))J_1(t_1), \quad t_1 < t_*$$

$$\text{where} \quad J_1(t_1) = \sup_{t\in[0,t_1]} \int_0^t \frac{(1+t)^{\frac{3}{2}} ds}{(1+t-s)^{\frac{3}{2}}(1+s)^{\frac{K}{2}-\frac{3}{2}-\sigma}} \le \bar{J}_1 < \infty$$

if $\frac{K}{2} > 3 + \sigma$. Hence, $K > 11$.

Similarly,

$$m_2(t_1) \le Cd_0 + C(m_1^2(t_1) + m_1(t_1)m_2(t_1) + m_2^{K-2}(t_1)), \quad t_1 < t_*$$

$$|m(t_1)| \le Cd_0 + C|m(t_1)|^2, \;\; t_1 < t_*, \;\; m(t_1) = (m_1(t_1), m_2(t_1))$$

Therefore, $|m(t_1)|$ is bounded for $t_1 < t_*$, and moreover,

$$|m(t_1)| \le C_1 d_0, \quad t_1 < t_*$$

since $|m(0)| \le \sqrt{2}d_0$ is small. The constant C_1 does not depend on t_*. We choose d_0 so small that $d_0 < \epsilon/(2C_1)$ and $|m(t)| \le \epsilon/2$.

Example

Theorem There exists $U(\psi)$ satisfying conditions **U1-U4**.

Consider piecewise quadratic functions:

$$U_0(\psi) = \begin{cases} \frac{1}{2}(1 - b\psi^2), & |\psi| \le \gamma \\ \\ \frac{d}{2}(\psi \mp 1)^2, & \pm\psi \ge \gamma \end{cases}$$

The condition $U_0(\psi) \in C^1(\mathbb{R})$ implies

$$b = \frac{1}{\gamma}, \quad d = \frac{1}{1-\gamma}, \quad 0 < \gamma < 1.$$

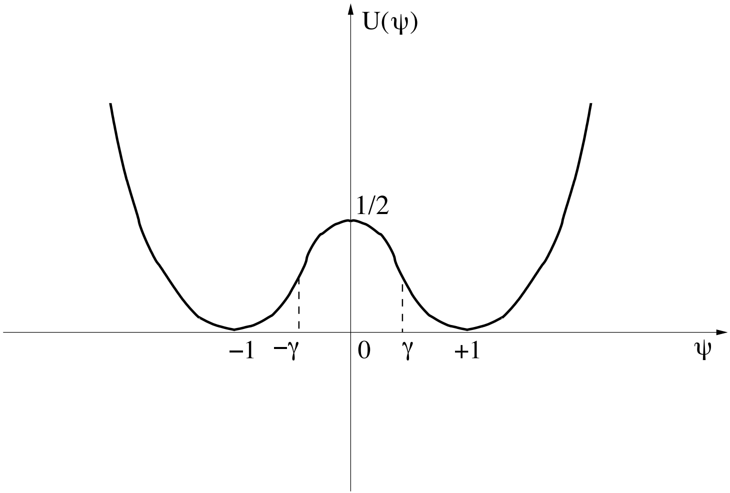


Figure: Potential U_0

The kink solution is such:

$$s(x) = \begin{cases} C \sin \sqrt{b} x, & 0 < |x| \leq q \\ sgn(x)(Ae^{-\sqrt{d}|x|} + 1), & |x| > q \end{cases}$$

where

$$C = \sqrt{\gamma}, \quad A = (\gamma - 1) \exp(\sqrt{\gamma/(1-\gamma)} \arcsin \sqrt{\gamma})$$

$$q = \sqrt{\gamma} \arcsin \sqrt{\gamma}.$$

The corresponding Schrödinger operator reads

$$H = -\frac{d^2}{dx^2} + W_0(x)$$

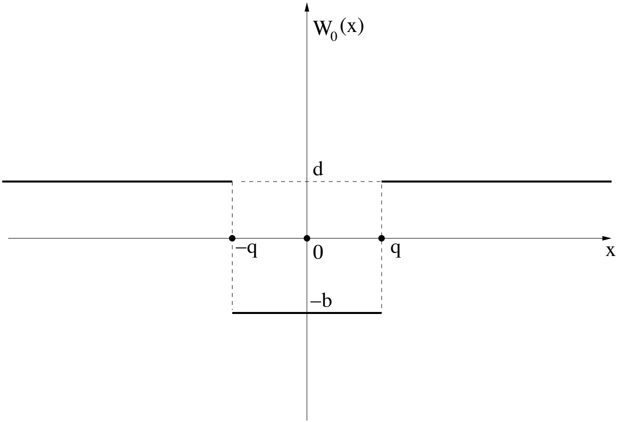


Figure: Potential W_0

$$W_0(x) = U_0''(s(x)) = \begin{cases} -b, & |x| \le q \\ d, & |x| > q \end{cases}$$

The continuous spectrum $\sigma_c = [d, \infty)$. The point 0 is the groundstate since it corresponds to the symmetric positive eigenfunction

$$\varphi_0(x) = s'(x)$$

Therefore, the discrete spectrum $\sigma_d \subset [0, d]$, and the next eigenfunction $\varphi_1(x)$ should be antisymmetric. Let γ_k, $k = 1, 2, ...$ be the solution to

$$\frac{\arcsin \sqrt{\gamma_k}}{\sqrt{1 - \gamma_k}} = \frac{k\pi}{2}$$

Numerical calculation gives

$$\gamma_1 \sim 0.64643, \quad \gamma_2 \sim 0.8579$$

$$\gamma_3 \sim 0.92472, \quad \gamma_4 \sim 0.95359, \quad \gamma_5 \sim 0.96856...$$

The set has a limit point 1

There is exactly one eigenvalue $\lambda_0 = 0$ if $\gamma \in (0, \gamma_1]$
There are exactly two eigenvalues $\lambda_0 = 0$ and $\lambda_1 \in (0, d)$ if $\gamma \in (\gamma_1, \gamma_2]$
etc.

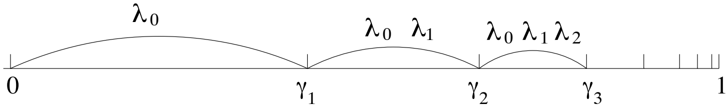


Figure: Spectr

The edge point d is resonance only if $\gamma = \gamma_k$, $k = 1, 2, ...$

The Fermi Golden Rule holds for all $\gamma \in (\gamma_1, \gamma_2)$ except for the one point $\gamma_* \sim 0.7925$.

Conclusion: If $\gamma \in (\gamma_1, \gamma_2) \setminus \gamma_*$ then $U_0(\psi)$ satisfies conditions **U1-U4**

We also have constructed smooth potentials $U_\epsilon(\psi)$ close to $U_0(\psi)$ satisfying conditions **U1-U4** for small ϵ

EK, A.Komech, S.Kopylov, Nonlinear wave equations with parabolic potentials, *J. Spectral Theory* (2013)

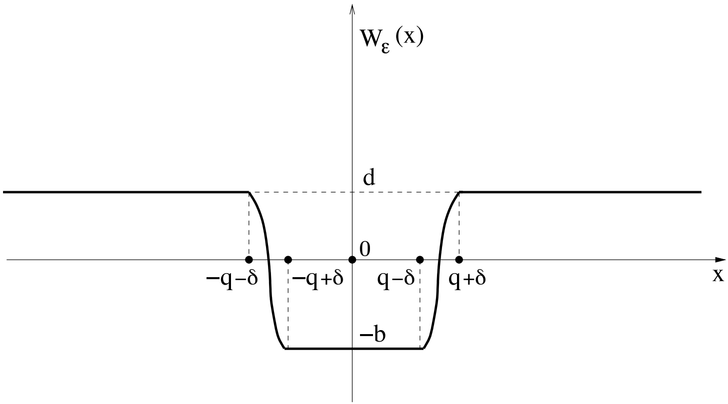


Figure: Potential W_ϵ

$\delta = \delta(\epsilon) \to 0$ as $\epsilon \to 0$