Contact interactions

Gianfausto Dell'Antonio

Dept.of Mathematics , U. Sapienza, Roma

and

Mathematics Sector , Sissa (Ts)

Lecture at Como

Sept. 2017

In this lecture we consider in the limit $\epsilon \to 0$ a system of $N \geq 3$ massive particles in R^3 which satisfy the Schrödinger equation with hamiltonian

$$H^{\epsilon}(V) = H_0 + \sum_{i,j=1...N}' V^{\epsilon}_{i,j}(|x_1 - x_j|) \qquad H^0 = -\sum_{k=1}^{N} \frac{1}{2m_k}\Delta_k \quad (1)$$

The sum runs over *a subset* Γ^c of pairs of indices and the potentials scale as $V^{\epsilon}_{i,j}(|y|) = \frac{1}{\epsilon^3} V_{i,j}(\frac{|y|}{\epsilon})$ where $V_{i,j}$ are functions of compact support.

In the limit $\epsilon \to 0$ these potentials are distributions supported by the contact (or coincidence) hyper-planes $\Gamma^c_{i,j} \equiv \{x_i = x_j\}$. We take the forces to be attractive.

We call $\Gamma \equiv \cup_{i,j} \Gamma^c_{i,j}$ *coincidence manifold*.

The limit, if it exists, is a self-adjoint extension of the symmetric operator H_0 defined as the free hamiltonian on functions with compact support and supported away from Γ.

We shall see that under a symmetry requirement and for a range of masses the limit exists in strong resolvent sense and characterizes uniquely a self-adjoint extension (among the infinitely many that can be defined) of H_0 restricted off Γ.

For other symmetries and/or for a complementary range of masses the limit exists in a weaker sense and the limit set is a family of self-adjoint extensions.

For example for N identical spin $\frac{1}{2}$ fermions the limit exists in the strong resolvent sense for all $N \in \mathcal{N}$ (unitary gas) .

For N identical boson the limit exists only in a very weak sense and the limit set is not one point.

One can select one extension by asking that the wave function of the bound state(s) have the strongest possible singularity.

This corresponds to the lowest possible energy (in physics this is justified by stability considerations).

We shall explain the strategy of the proof and give details for simple cases.

We call *contact interaction* the limit self-adjoint operator if it exists.

Notice that contact interaction *is not defined* for two isolated bodies; contact interaction should not be confused with *point interaction* as defined by S.Albeverio et al [A].

Some of the experimental data are obtained for systems in which a zero energy resonance is monitored by applying suitable magnetic fields (Fesbach resonances) [W,C].

We shall see that for $N \geq 3$ contact interactions can be defined also in presence of two-body zero energy resonances.

Recall that a zero energy resonance for a two-body potential invariant under rotation is defined, in relative coordinates, as a rotational invariant solution $\phi(x)$ of $(\Delta + V)\phi(x) = 0$, $x \neq 0$ which is only locally in L^2.

One has $\phi(x) = \frac{a}{|x|} + \psi, \ \ a, \in R\psi \in L^2$. $4\pi a$ represents the *scattering length* .

One has then $\phi(x) = a \int_{S^3} d\hat{k} \int_0^\infty d\rho \frac{e^{i\hat{k}.\hat{x}}}{\rho}$

Notice that while a resonance does not belong to $L^2(R^3)$, pairs of resonances belong to $L^2(R^6)$ *provided one excludes from the domain the set* $\hat{k}_1 = \hat{k}_2$ (parallel tails") .

In particular if the wave function is antisymmetric with respect to the interchange of a set of coordinates, the wave function with two zero-energy resonances is in $L^2(R^6)$.

We consider only this case.

We denote by H_F resulting operator; its construction is similar to the construction of the Friederichs extension in $L^2(R^3)$ i.e *eliminate first* the zero energy mode(s) , and *then* construct the laplacian [F].

In absence of resonances we use the notation H_M for the closure of the free hamiltonian H_0 defined on functions of support compact and away from coincidence planes.

This operator has been used by Minlos [M1][M2] in his analysis of contact interactions for $N = 3$.

When there is no need to distinguish between the two operators, we will use the symbol $\tilde{H}$

For $N \geq 3$ the presence of zero energy two-body resonances does not lead to difficulties in the Birman-Schwinger-Krein formula for the difference of resolvents.

The difference between resonant and non resonant contact interaction is seen when there are bound states in the three and four body channels.

For identical fermions we exclude "parallel resonances" and therefore H_F is smaller than H_M and the bound states for the case of resonances have *lower energy* .

One has by construction $H_F = (1 - P)H_M$ where P is the projection onto the subspace that is omitted in the definition of H_F . Since P commutes with the spectral projections of H_M the two operator commute and the extension of the restrictions have the same eigenvectors if the point spectrum is simple.

But the eigenvalues of the total hamiltonian differ.

The difference of the spectra is small if the bound states are sharply localized, but it may become relevant if their essential support is large, as is sometimes the case .

There are reports ([C,M,P] [Ba, P]) of a discrepancy between experimental results and theoretical predictions.

Since the theoretical analysis refers often to the operator H_M (because its Green function has a simpler form) this discrepancies may be a point in favor of H_F i.e of contact interaction with two-body zero energy resonances.

In the classical case constraints on the behavior at some boundary are attributed to strong forces acting at the boundary.

In Quantum Mechanics contact interactions are often defined by imposing boundary conditions at the coincidence manyfold $\Gamma \equiv \cup_{i,j}\Gamma_{i,j}$.

The boundary conditions are expressed by the requirement that at the boundary functions in the domain have at most the singularity

$$\phi(X) = \frac{C_{i,j}}{|x_i - x_j|} + b_{i,j} + 0(|x_i - x_|) \quad i \neq j \tag{2}$$

where the constants $C_{i,j}$ characterize the boundary conditions

.

Conditions of this type were used already in 1935 by H.Bethe and R.Peirels [B,P] in the description of the interaction between proton and neutron. In that case there are no Y variables.

These condition for $N = 3$ were later used by Skorniakov and Ter-Martirosian [S,T] in their analysis of three body scattering within the Faddaev formalism. For every N we shall call them Bethe-Peirels (B-P) boundary conditions .

When $b_{i,j} = 0$ (the only case we consider) the boundary condition can be described formally by potentials $v_{i,j}$ that are *distributions supported by the boundary*.

This step is crucial if one wants to see contact interaction as limits of interactions with smaller and smaller range.

The equivalence can be seen by taking the scalar product with a function in the range of the free hamiltonian and integrating by parts. This is the same procedure used to define the different realizations of the laplacian on $(0,+\infty)$

In Theoretical Physics the interest in the subject was renewed by the flourishing of research on ultra-cold atoms interacting through potentials of very short range in which a zero–energy resonance can be induced by a magnetic field (Fesbach resonance).

From the mathematical point of view the problem of contact interaction between three particles was analyzed by R.Minlos, [M1,M2] concentrating on (the physically relevant) case of two fermions of mass one interacting, through zero range forces (contact interactions) , with a particle of mass m.

There is general strategy [v N] to find self-adjoint extension of a symmetric operator A: for the given operator A one considers then the graph of $A^* + i$ (with the graph norm) and finds the extensions in the direct sum of Hilbert spaces.

Considering the graph gives a control at the same time of the domain of A^* and of its range. Recall that for a self-adjoint operator one has $A = A^*$.

In our case we have a natural positive self-adjoint extension $\tilde{H}$ and we want to impose contact boundary conditions.

We are therefore looking for extension *of the restriction* [P] of $\tilde{H}$ to functions that are supported away form the hyperplanes on which we want to impose boundary conditions.

Contact interactions are *a special class* of extensions of $\tilde{H}$ defined away from the contact manifold.

Since $\tilde{H}$ is positive a standard way [B][K] to find extensions would be to consider their action on the range of $\tilde{H}+\lambda$ where λ is an arbitrary positive number.

We consider instead a quadratic form version i.e. we introduce the map $\phi \to \psi \equiv (\tilde{H}+\lambda)^{-\frac{1}{2}}\phi$ where λ is an arbitrary positive number

The map is a compact immersion of the *physical space* into a space that we call *Krein space*.

It could be called *space of charges* ; the Krein map has the same function as the map from potentials to charges in electrostatics.

The compactness of the immersion is used to transfer strong closure from Krein space to weak closure in ”physical space” and then to strong closure if the form is positive and closable.

On the domain of the operator in Krein space the map to physical space is obtained (by duality) by applying the Krein map to the wave function.

This allows us to obtain results, as we will see, with a minimal effort.

Remark that these extensions *are not* all self-adjoint extensions that one can obtain of the free hamiltonian restricted to function that have support outside the coincidence hyperplanes.

Birman's theory (and von Neumann theory) allow for a multitude of self-adjoint extensions.

For example for $N = 4$ Minlos and Faddeev [M,P] have constructed a self-adjiont extension *unbounded below* (the system considered is one scalar particle interacting with three identical particles).

In this case the quadratic form has a stronger singularity at coincidence points (as $\frac{1}{|x_i-x_j||x_i-x_k|}$) and cannot be obtained as weak limit from regular potentials.

The use of quadratic forms for the description of singular interactions has a long history.

It is consistently used in [F] and in Simons' and in Kato's books.

In the context of contact interaction it appears first in Danilov [Da] but Kushmanenko [K] attibutes it to Faddaev.

Quadratic form for the unitary gas model were studied in [F,T]

The Krein map is used in [M1] [M2] although not explicitly mentioned.

The analysis in [C1] and [C2] makes use consistently of quadratic forms techniques.

We construct these special self-adjoint extensions using *Krein space* as auxiliary space. In this space convergence of the quadratic forms when $\epsilon \to 0$ is more easily established.

We call *Krein map* the (compact) embedding of *physical space* $L^2(R^{3(N-1)})$ into Krein space.

The embedding is obtained through the operator $\sqrt{\tilde{H}+\lambda}^{-\frac{1}{2}}$.

In an appendix we provide the relation between the auxiliary space we use and the method of "heat kernel renormalization" frequently used in Theoretical Physics.

One can compare this procedure with the one in electrostatics that leads from the description through *potentials* to that by means of *charges* ; the Krein map plays the role of the map from potentials to charges and the Krein space plays the role of space of charge distributions.

Notice that also in electrostatics it is easy to define distribution of charges on a co-dimension one manifold but the definition of point charge requires more care (as is the case for the definition of point interaction).

The case in which there is a family of extensions should be compared with the case in which the bounary has sharp concave edges or horns.

Definition

We define *contact interactions those extensions of the restriction* [P] of $\tilde{H}$ to functions with support away from the contact manyfold which are obtained through the Krein map. .

We define *resonant contact interactions* those that are obtained form H_F.

Contact interactions represent boundary conditions at the manifold Γ.

In Krein space for a range of parameters these conditions are more singular than those suggested by the B-P apprach.

In fact the leading boundary behavior may be $|x_1 - x_j|^{-\alpha}$ with $1 < \alpha < 2$ and also $|x_i - x_j|^{-2} log(|x_1 - x_2|)$

In Krein space the N-body system is described by a quadratic form which is the difference of two positive quadratic forms.

The first is the quadratic form of the operator $\sqrt{\tilde{H} + \lambda}$ (the image in Krein space of $\tilde{H} + \lambda$) and the second is the convolution of the kernel of $\frac{1}{\tilde{H}+\lambda}$ with the boundary potentials (the image of the potentials).

The positive parameter λ is introduced to make the Krein map compact; it can take an arbitrary positive value.

For $\lambda = 0$ the forms are homogeneous of order one.

For a range of masses (that depends on the symmetries of the system) the resulting form is positive and defines a self-adjoint operator (in Krein space).

For the complementary range the form is not positive nor closed; it defines a family of self-adjoint extensions in Krein space. We call this the singular case.

We will recognize this as a Weyl limit circle phenomenon.

The forms of the different extensions differ from each other by a form that represents in Krein space a different behavior at triple coincidence point (where the eigenfunctions are singular).

Each of the extensions has the positive real axis as (absolutely) continuum spectrum and has a non degenerate point spectrum; the latter may consist of infinitely points diverging geometrically to $-\infty$.

In order to have the description in "physical space" one must invert the Krein map.

If the limit form is positive and closed this is done exploiting the compactness of the Krein map.

The domain of the resulting self-adjoint operator is obtained (by duality) acting with the Krein map on the domain of the operator in Krein space.

The structure of the Krein map implies that in this "regular case" the functions in the domain of the operator in physical space satisfy the B-S boundary conditions.

Convergence when $\epsilon \to 0$ *does not take place* but there is an inferior limit .

When limit form is not closed in order to find the "lift" to physical space one cannot consider separately the members of the family of form is not a direct sum of forms in different spaces

One rather relays on $\Gamma-$ convergence [Br].

There is therefore in physical space *a distinguished extension* (obtained by Γ-convergence) that corresponds to minimal energy (and maximal singularity of the wave functions of the bound states).

According to Γ- convergence this extension is obtained by *sequential convergence*.

The corresponding self-adjoint operator is the limit the sequence $H_0 + V^\epsilon$

As a result of the lift, each eigenfunction is *flattened* and geometric convergence to $-\infty$ of the (energy) spectrum is turned into geometric convergence to zero (Efimov effect).

Notice that this Efimov effect has nothing to do with the Efimov effect due, in case of regular potentials, to the presence of zero-energy resonances in at least two channels in a three body problem.

I fact, in the present case, this effect *is independent of the presence of two-body resonances*.

Remark

The "tail" of the Efimov states (small energy states) are hardly present in a realistic model (which keeps into account other type of interactions and corresponds to $\epsilon \neq 0$).

What can be seen in experiments are a few members of the "head" (low lying states) of the Efimov sequence; they should be recognized as Efimov states for the geometrical scaling property.

This states are less affected by the other short range interactions.

On this states the effect of the zero-energy resonances (hamiltonian H_F instead of H_M) might be more visible.

There body Efimov states have probably be seen experimentally [Pe], [C,M,P] and also four-body Efimov states have been reported [Ba,P].

We now describe briefly the cases $N = 3$ and $N = 4$ and then concentrate on two cases:

1) a pair of identical scalar fermions of unit mass interacting with a third particle of unit mass.

2) a pair of identical spin $\frac{1}{2}$ fermions in contact interaction .

In both cases the hamiltonian is positive.

The last case generalizes without difficulty to the case of N pairs of identical spin $\frac{1}{2}$ (unitary gas). Also for this system the hamiltonian is positive.

We already remarked that for general values of N the quadratic form which represents the extension in Krein space is the difference of two positive quadratic forms, both closed: the form of $\sqrt{\tilde{H} + \lambda}$, image under the Krein map of the kinetic term, and the convolution of $(\tilde{H} + \lambda)^{-1}$ with the image of the distributional potential at the boundary Γ.

The latter is an L^1 function, limit in the strong L^1 topology of the image under the Krein map of the ϵ -dependent potentials .

For a range of masses and given symmetries, the resulting form in Krein space defines uniquely a self-adjoint operator in "physical space" .

This operator is the limit in strong form convergence sense of the interaction hamiltonians for regular short range potentials.

For other symmetries and range of masses the form is associated in physical space to a family of self-adjoint operator all bounded below and with negative point spectrum accumulating at zero (Efimov effect [E]).

We shall see that for these self-adjoint extension the analysis of the negative spectrum of the N-body problem *can be reduced to the analysis of the three- and four-particle subsystems*.

We shall prove that the negative part of the spectrum of each extension for an N body system is a point spectrum and is contained *in the algebraic union* of the point spectra of its three- ant four-particle subsystems.

In particular a system composed of N spin $\frac{1}{2}$ fermions is stable (its spectrum is the positive real axis).

,

We shall consider first the case of the three-body system in the non resonant case.

We shall give details only for the physically relevant case of two identical fermions of unit mass in contact interaction with a particle of mass m.

We shall then analyze briefly the case in which the identical particles are bosons.

According to the general prescription, the kernel of the "potential" in Krein space is , in the center of mass reference frame

$$-\frac{1}{p^2+q^2+\alpha p.q+\lambda} \quad p,q \in R^3 \quad \alpha = \frac{2}{1+m} \tag{3}$$

This quadratic form and its restriction to the case in which two of the particles are identical fermions or identical bosons has been put in diagonal form by R.Minlos [M1], [M2] (see also [C1], [C2]) by using a Mellin transform.

This explicit diagonal form permits to obtain detailed information for all m about the self-adjoint extensions and their spectral properties both for the fermionic case and for the bosonic case.

For a range of the parameter m(different for bosons and fermions) there is a family of self-adjoint extension parametrized by $0 < \alpha \leq 1$ and each of them may have a finite or infinite number of negative eigenvalues (accumulating to $-\infty$).

It is proved in [M1][M2] that for each value l of the angular momentum there is an interval $C_l^* < C \leq C_l^{**}$ the symmetric operator E_m^l is in the Weyl limit circle case and has a one parameter family of self-adjoint extensions $E_{\alpha,m}^l \quad \alpha \in (0,1]$ with a single eigenvalue μ_l^α which is a strictly decreasing function of α.

When $0 < C \leq C_l^*$ each symmetric operator E_m^l is still in the Weyl limit circle case but each of the extensions $E_{\alpha,m}^l$ has an infinite number of bound states with eigenvalues $\mu_l^{n,\alpha}$ which diverge to $-\infty$ geometrically scaled.

Each eigenvalue is non degenerated, *the spectra are disjoint for different values of the parameter* α and $\cup_{\alpha\in(0,1]} = (-\infty, 0)$

The eigenfunctions are known explicitly .

Since for $N \geq 3$ the diagonal form is not available it is convenient to write the "potential term" in a form convenient to an analysis in configuration space.

This will help also in a better understanding of the nature of the spectrum.

One has then

$$-\frac{1+m}{m}\frac{1}{(p-q)^2} + \Xi(p,q,\lambda) \tag{4}$$

where Ξ is a positive kernel with $\Xi(p,p,0) = 0$

Therefore in Krein space the operator is

$$2\pi^2 \frac{m}{m+1}\sqrt{-\Delta} - \frac{4\pi(1+m)}{m}\frac{1}{|x|} + \tilde{\Xi} \tag{5}$$

where $\tilde{\Xi}$ is a positive operator represented with smooth kernel.

In the case of the fermions one must first anti-symmetrize the potential form and then write the kernel as function of the variable $p-q$ and $p+q$.

The resulting symmetric operator is

$$2\pi^2 \frac{m}{m+1}\sqrt{-\Delta} - \frac{1+m}{8\pi(2m+m^2)}\frac{1}{|x|} + \tilde{\Xi}' \tag{6}$$

where Ξ' is a positive operator with locally bonded kernel.

Notice the the coefficient of the second term diverges as $m \to 0$ both for fermions and for bosons.

It follows that the form in Krein should be compared with the quadratic form of the symmetric operator

$$\sqrt{-\Delta} - C(m)\frac{1}{|x|} \tag{7}$$

where $C(m)$ is a positive function of the parameter m,

This function is different in the bosonic and in the fermionic case but in both cases increases monotonically to $+\infty$ as $m \to 0$.

Symmetric operators of the form $\sqrt{-\Delta} - \frac{C(m)}{|x|}$ have been studied extensively (see e.g. [IJ][B,R]) as a function of $C(m)$, originally in the context of the non relativistic hydrogen atom.

We denote them by H_R (relativistic atom).

For these operators there are threshold values C^*, C^{**} such that for $C > C^{**}$ the spectrum is absolutely continuous and positive.

For $C^* < C \leq C^{**}$ there is a continuous family of self-adjoint extensions, each with a negative eigenvalue, and for $=< C \leq C^*$ the negative spectrum is pure point and accumulates geometrically to $-\infty$ (a Weyl limit circle effect) .

In the latter case the eigenfunctions concentrate at the origin (eigenvalues and eigenfunctions are known explicitly)

These results can be transcribed in function of the mass m of the third particle and lead to different mass intervals in the bosonic case and in the fermionic one.

In particular denoting by M_l^*, M_l^{**} the mass thresholds in the fermionic case one has $1 > M_l^{**}$ for every l whereas in the bosonic case one has $1 < M_0^*$.

Since the eigenfunctions are known it is possible to show that the operators $\tilde{\Xi}'$ and $\tilde{\Xi}$ are small on the eigenvectors of the self-adjoint operators associated to $\sqrt{-\Delta} - \frac{C(m)}{|x|}$.

In the case $m < M^{**}$ one can use rearrangement inequalities and regular perturbation theory with convergent series to prove that also for the complete forms the same multiplicity of self-adjoint extensions occurs with the same thresholds holds.

This approach has the merit to point out the role of the limit circle property and does not rely on an explicit diagonalization, which is not known for $N > 3$.

We now consider briefly the case of two identical bosons of mass 1 and a third particle of mass m.

In the case of two identical boson in Krein space the part of the energy form that comes form the boundary conditions is

$$Q_2(p,q)' = \frac{-\frac{2}{1+m}(p.q)}{(p^2+q^2)^2 - c_f(m)(p.q)^2} \qquad c_f(m) = \frac{4}{(1+m)^2} \tag{8}$$

Now the quadratic form is the sum of a positive form Ξ'' and of $\sqrt{\Delta} - \frac{C_b(m)}{|x|}$ where $C_b(m) > C_f(m)$ (b for bosons).

Again one can introduce a "relativistic atom" comparison hamiltonian.

It is convenient here to notice that both the given form and the relativistic atom operator are invariant under rotations and can be decomposed in angular momentum states.

Notice that the expectation value of $\sqrt{-\Delta}$ depends on the angular momentum of the state.

One can again derive the properties of the spectrum from those of the spectrum of H_R.

Denote by M_j the value of m such that for $m \leq M_j$ the expectation value of the component of the quadratic form with angular momentum j the form associated to $\sqrt{\Delta} - \frac{C_b(m)}{|x|}$ ceases to be positive.

One verifies that $1 < M_0$ whereas $M_j > 1$ for all $j \geq 1$

Again this result is also true for the complete form.

Therefore for two identical bosons of mass 1 which do not interact among themselves and are in contact interaction with a particle of the same mass 1, in the zero angular momentum sector there are infinitely many extensions.

In Krein space their negative spectrum is pure point and not degenerate and is unbounded below with asymptotically a geometric rate.

There is no interaction in the $l = 1$ channel. The quadratic form is positive for $l \geq 2$.

One must now recall that this analysis is done in Krein space, and to draw conclusions relevant for physics one must come back to physical space inverting the Krein map.

In our case the Krein map is a change in metric from L^2 to $\mathcal{H}^{-\frac{1}{2}}$.

We consider first the case in which the form in Krein space is positive. Since the map preserves positivity, the image in physical space is positive.

In Krein space the form was strongly closed. By compactness of the Krein map, inverting the map one obtains a form that is weakly closed.

But the form is positive and weakly closed and therefore it is also strongly closed. It determines therefore a positive self-adjoint operator in the physical space $L^2(R^6)$ in the center of mass frame); this is the extension we are looking for.

Consider now the case in which there is in Krein space a family of self-adjoint operators each of which has a continuous positive spectrum and one or more isolated negative bound states.

The positive part of the spectrum leads as before to a positive self-adjoint operator.

As for the point spectrum, by duality the inversion is achieved on each eigenfunction by an application of the Krein map.

The eigenfunctions have now larger support and the eigenvalues are changed.

The sequence of negative eigenvalues that diverges geometrically to $-\infty$ is turned in a sequence that converge geometrically at zero (Efimov effect).

In the Physical literature the bound states associated to the singularity in pairs of particles are called *quadrimers* and the ones associated to the three-body singularities are called *trimers*.

There are reports that both have been seen in experiments. [C,M,P] [Ba,P] .

Recall that if there are resonances the reference operator is H_F and this implies that that the term which we have called "kinetic energy" is somewhat smaller and the bound states have lower energy.

We consider now the case of four particles.

Again in order to study this system we introduce an auxiliary space, this time by means of the operator H_M for the four particle systems.

We assume here that there are enough symmetries so that the kernel of the potential term is a function of two coordinates.

There are now several ways to write (for $\lambda = 0$) *the denominator* as sum of the squares of differences between sets of coordinates.

One way is to isolate a coordinate of one particle and proceed with the others as for $N = 3$.

The other is to consider the square of the difference of the coordinates of the barycenter of two interacting pairs.

This contribution to the quadratic form can be interpreted (formally) as due to the (Coulomb) interaction between the charge distributions of distinct pairs of particles.

The singularity in configuration space are the same type as the one from a three-body subspace *but in different position in configuration space*.

The operator in Krein space is now

$$\sqrt{H_M} - \sum_{i,j} \left(\frac{C_{i,j}}{|x_i - x_j|} + \tilde{\Xi}_{i,j} \right) \tag{9}$$

where the operator $\Xi_{i,j}$ has locally bounded kernel, x_j are the positions of pairs in contact and $C_{i,j}$ are positive numbers that depend on the masses of the particles and on the symmetries of the system.

For a range of masses the sum of the "potential part" and the "kinetic energy" part is positive.

Upon inversion of the Krein map the form is only weakly closed but since it is positive strong closure follows. In this case the form defines a positive self-adjoint operator in physical space.

For another range one or both singular term dominate and the operator is in the limit circle case.

It has a family (generically a two-paramter family) of self-adjont extensions with continuous positive spectrum and with negative point spectra.

The inversion of the Krein map is done as in the case $N = 3$ i.e for the continuous part by using the compactness of the Krein map and for the discrete part action on the eigenfunctions.

By iteration this is true for a N-particle system.

If x_j x_k represent the position of two pairs in contact interaction, the potential part of the quadratic form can be written in configuration space for a suitable choice of coordinates

$$U = -\sum_{i,j} \frac{C_{i,j}}{|x_i - x_j} + W(x_i, x_j) + \lambda \quad C_{i,j} > 0 \tag{10}$$

for some pairs of coordinates.

The operator W has a locally bounded kernel and therefore the sharp minima are isolated. This is a consequence of the absence of triple contact points.

Depending on the masses and the symmetries the symmetric operator may be in the limit circle case in one or more variables.

Since the singularities occur in different positions in configuration space , the quadratic form in Krein space has a family if extensions (that may depend now on several parameters) each of which has the positive real line as positive (continuous) spectrum and with a negative point spectrum that may be finite or infinite.

In the latter case the eigenvalues converge to $-\infty$.

Again, to the positive part the spectrum corresponds to a positive quadratic form and when lifted to the physical space is a positive self-adjoiny operator.

For the negative part, the inversion of the Krein map is done separately for each extension and each eigenvector of that extension.

Due to the smoothing properties properties of the inverse of the Krein map in configuration space for a range of masses one has the Efimov effect in physical space.

We give now a more detailed analysis of the case of two pairs of identical spin $\frac{1}{2}$ fermions.

This form in Krein space representing the potential part of two pairs of particles in contact interaction can more easily be expressed in Fourier transform (R.Minlos , private communication). It is a sum of three terms C_0, C_1, C_2

$$(\phi, C_0\phi) = 2\pi^2 \int dkds\bar{\phi}(k,s), \sqrt{\frac{3}{4}(k^2+s^2)+\frac{1}{2}(k,s)}\phi(k,s)$$

$$(\phi, C_1\phi) = \int dkdsdw\bar{\phi}(k,w)\frac{\phi(k,w)+\phi(k,s)}{k^2+s^2+w^2+(k,s)+(k,w)+(s,w)}$$

$$(\phi, C_2\phi) = -\int dwdsdk\frac{\bar{\phi}(k,s)\phi(w-\frac{k+s}{2}, -w-\frac{k+s}{2})}{w^2+\frac{3}{4}(k^2+s^2)+\frac{1}{2}(k,s)} \qquad (11)$$

C_0 originates from the lift to Krein space of the kinetic energy, C_1 originates from the pair of two-body contact interaction between two pairs of particle and C_2 originates from the contact interaction of a triplet of particles (in presence of the fourth one).

Notice that C_1 is zero on functions that are antisymmetric under interchange of the two arguments. On these functions $C_0 + C_2$ is the integral with some positive weight of forms of the same type as encountered in the three-body case.

Therefore it is positive if the masses are equal.

On functions that are symmetric under interchange C_2 is positive and larger that half of the absolute value of C_1.

On these functions $C_0 + \frac{1}{2}C_1$ is positive since it is the integral with a positive weight of kernels that were analyzed in the case $N = 3$ and proved to be positive.

Therefore $C_0 + C_1 + C_2$ is the kernel of a positive operator. We have therefore proved that the operator associated to a system two pairs of identical spin $\frac{1}{2}$ fermions interacting through contact interactions is a positive self-adjoint operator in Krein space.

Since the Krein map preserves positivity also the corresponding operator in $L^2(R^9)$ is a positive self-adjoint operator (this result had previously been predicted [M,P] with the aid of a computer) .

We have done our analysis for the non resonant case. But it is easy to see that the resonant case (using H_F) leads to the same results.

As for the N-body problem we remark that we can consider only those particles that take part in contact interaction.

For the 4-particle system some contributions come from a three body problem.

Other contributions can be described in Krein space as due to a first order differential operator describing *a closed system* (two charges interacting through a potential field).

The singular part of this contribution does not change if the four particle system is regarded as subsystem of a five particle system.

We have seen that a system of two pairs of identical spin $\frac{1}{2}$ fermions in contact interaction is stable, i.e its hamiltonian is positive.

In the same way one proves that the hamiltonian of an arbitrary number of pairs of $\frac{1}{2}$ equal mass fermions is positive.

In the Theoretical physics literature this system is called *unitary gas*. We have therefore

Proposition The unitary gas is stable.

....

Proceeding as above one proves that for contact interactions of N bodies the (negative) point spectrum is completely determined by the three- and four -body subsystems.

Graphically one can describe this by saying that the contact interaction among N particles can be visualized by a diagram in which there is a link between two points if there is contact interaction between them.

The diagram is made of V shaped components, corresponding to the configurations in which one body can be in contact interaction with two other bodies, and an H shaped components representing Coulomb interaction between two pairs each in contact interaction.

For every N *in Krein space* the system is described by a quadratic form which is the sum the square root of the kinetic energy and of negative quadratic forms composed according the diagrammatic rule given above.

Depending on the masses and on the symmetries the form is either closed and semi-bounded or else the corresponding operator is in Weyl limit circle case.

This corresponds to families of self-adjoint extension with negative point spectrum.

These operators may be unbounded below in Krein space.

One finds the description in physical space by inverting the (compact) Krein map on the continuous part of the spectrum and for the negative point part (by duality) applying he Krein map separately for the eigenvectors. This makes the eigenfunctions more regular.

The resulting operator is bounded below for any N and any choice of masses.

In case the negative spectrum contains an infinite number of points, the Efimov effect occurs.

The conclusion one can draw in general is that the spectrum of the N body systems in Krein space is contained in the *algebraic sum* of the spectra of its three- and four-body components.

Something more can be said in special cases. For example in the case of contact interaction of a system composed of a particle of mass m and $N \geq 3$ identical scalar fermions of mass one, the spectrum is independent of N.

Another favorable case is the one in which all subsystems are stable.

We will prove that this is the case for a system of N pairs of spin $\frac{1}{2}$ identical fermions (unitary gas).; in fact we will prove that all three and four particles subsystems are stable.

From the analysis above it follows that there can be a three or four particle Efimov effect but there is no five body Efimov effect.

If there are other interactions this last statement must be taken with care.

For example two pairs of fermions can be bound *due to other short range forces* and the two *resulting bosons* interact with the other particle through a contact interaction.

For an outlook on experimental and theoretical results on the three and four body problem one can consult [C,M,P] [C,T] [Pe].

Remark

For regular potentials , pairs of resonances contribute *additively* to the resolvent.

This is true also for resonant contact interactions (as expected, due to strong or weak convergence)

Therefore also in the resonant case the negative part of the spectrum of the self-adjoint operators describing the system *is completely determined by the two- three and four-body subsystems.*

Notice however that in the resonant case the lower part of the spectrum of the operator that describes contact interaction is somewhat changed and this slightly modifies the energy of the bound states the Efimov effect.

.....

Contact interactions as limits

We define "regular" the case in which the form in Krein space defines uniquely a self-adoint operator and "singular" the other case.

Proposition

For $N \geq 3$ and in the regular case resonant contact interactions are the limit when $\epsilon \to 0$, in the sense of strong convergence of quadratic forms, of Schödinger operators with two-body potentials $V_\epsilon^{i,j}(|x_i - x_j|))$ *which have* a zero-energy resonance *and scale as* $V_\epsilon^{i,j}(|y|) = \frac{1}{\epsilon^3} V^{i,j}(\frac{|y|}{\epsilon})$ *so that* $\|V_\epsilon^{i,j}\|_1 < C$.

We first prove strong form convergence in Krein space.

In this space the potentials and their limit are quadratic forms obtained by convolutions the boundary potentials (distributions) with $(\tilde{H}+\lambda)^{-1}$ where $\lambda > 0$ (this parameter enters in the definition of Krein space and is arbitrary).

In Krein space the approximating potentials are negative L^1 functions uniformly bounded in L^1 norm and converge in L^1 to the boundary potentials.

Therefore as quadratic forms the approximating hamiltonians converge strongly to the quadratic form that describes contact interactions.

In the regular case, the limit form defines uniquely a positive operator in Krein space and therefore the approximating hamiltonians converge in strong resolvent sense to this positive operator.

Inverting Krein's map, one has strong resolvent convergence in physical space.

In the singular case the limit form does not define uniquely a self-adjoint operator.

It rather corresponds to a family of self.adjoint operators depending on a parameter $\alpha \in (0, 1]$.

According to Weyl's theory each extension has one or infinitely many eigenstates.

The Krein map can be used in other problems.

A model example is the Salpeter (semi-relativistic Schrödinger) equation with an attractive delta potential.

The two-body case is analyzed in [A,K] as point perturbation of a pseudo-differential operators .

In [A,K] it is proved that the symmetric operator $\sqrt{-\Delta + 1}$ defined on smooth functions with compact support away from the origin has deficiency indices $(1, 1)$ and therefore admits a one parameter family of self-adjoimt extensions.

The Authors give explicitly the deficiency subspaces (and also the scattering structure).

There is one family of self-djoint extensions; the "boundary conditions" at the origin are now

$$\psi(x) = c_\psi log|x| + \psi(0) + o(1) \qquad c_\psi \in C \tag{12}$$

One can perform the analysis of the N-body problem of the Salpeter model with contact interaction following the scheme used for the Schrödinger case.

The boundary behavior is more regular but now the differential operator is of first order.

The Krein map is given now by the operator $(\bar{H} + \lambda)^{\frac{1}{4}}$

It has as usual the effect of smoothing the distributional "potential" at coincidence hyperplanes but at the same time it lowers the differential structure of the positive part which is now a (pseudo)-differential operator of order $\frac{1}{2}$.

In Krein space the quadratic form is now

$$(\phi, (\bar{H} + \lambda)^{\frac{1}{4}}\phi) - (\phi, \Xi_\lambda \phi)) \tag{13}$$

where the operator Ξ the convolution of $(H_M + \lambda)^{-\frac{1}{2}}$ with the delta distributions at the coincidence hyperplanes.

Depending on the choice of the masses this quadratic form is positive or it has a negative term with a logarithmic singularity.

In the latter case the form is not closed and the symmetric operator can be decomposed in a family of self-adjoint operators with simple negative point spectrum.

The eigenvalues may be infinite in number; in this case they accumulate geometrically to $-\infty$.

Also here on goes back to physical space undoing the Krein map. This is done by compactness on the continuum part of the spectrum and separately on each eigenvector for the discrete part.

Once more the sequence of eigenvalues shows the Efimov effect.

It would be interesting to do the same analysis for a system of non relativistic particles described by the Pauli hamiltonian

Appendix 1: Krein space map vs. heat kernel renormalization.

For $N \geq 3$ we used the Krein map to determine the properties of the quadratic forms *in Krein space* and then of the self-adjoint operators in physical space. .

In case of multiplicity of extension this multiplicity was recognized from the structure of the quadratic form.

The Krein method has its justification in the theory of self-adjoint extensions and it not a perturbative method.

An analysis of contact interactions is often done in Theoretical Physics [E,T] by a *renormalization procedure* somewhat similar to the steps that lead to the definition of a point charge in electro-statics.

In[E,T] for contact interactions in the N-body case, $N \geq 3$, in order to avoid divergences in a perturbative analysis, the interaction is modified by convolution with a Kernel, usually a heat kernel, which depends on a parameter ϵ.

The leading term, divergent term in the result is then neglected. This procedure is called *heat kernel renormalization*

We now prove that the Krein map provides the same result as the heat kernel renormalization.

If H_0 is the self-adjoint operator that represents to free hamiltonian and λ is an arbitrary positive number one has

$$\int_0^\tau e^{-(H_0+\lambda)t} dt = \tau + \frac{1}{H_0+\lambda} + O(\frac{1}{\tau}) \tag{14}$$

The heat kernel regularization consists *by definition* in taking the limit $\tau \to \infty$ neglecting the linearly divergent constant.

The need of renormalization can be viewed as due to the fact the "correct" wave function is too singular to be obtained by a perturbative analysis.

The introduction of the heat kernel is artificial, but it allows to "compress " the divergent term in a multiplicative factor.

It follows from eq. (19) that in the limit $\tau \to \infty$ the prescription of the heat kernel renormalization scheme is to modify all expectation values by inserting the operator $\frac{1}{H_0+\lambda}$.

In the quadratic form version this is precisely the definition of the Krein map.

Of course *to obtain result in physical space one must invert the Krein map*.

From a formal point of view, using Krein space allows the solution to be in a larger space (made of less regular functions).

Typically the solution found by perturbation theory will be in the smaller space to first order but the further terms belong to the larger space *and not to the smaller space*(the "boundary potential" acts as a function in the larger space and a distributions in the smaller space).

Therefore in this case perturbation theory is not even defined in the smaller space.

The perturbation series can be often summed in the larger space; the solution obtained (to the initial value problem or for the quadratic form that represents the operator) belongs to the larger space.

One must now lift the quadratic form (and the solution) to "physical space". This is a non perturbative map.

If the self-adjoint extension is unique, this can be done in an unique way.

This means that the solution belong to the smaller space, although the perturbation expansion *is only formal in this space*.

In the N-body problem this occurs for a range of the masses. Outside this range there are infinitely many self-adjoint operators.

As in the two-body case, more empirical means (i.e experimental data) should be used to choose "the right one".

Further remarks

A system of N identical bosons in the limit $N \to \infty$ is often described through a non-linear (cubic) Schrödinger equation.

This equation is meant to describe the dynamics of the one-particle marginals for a system of a very large number N of identical particles interacting *through a two-body potential* with a given scattering length,

If the gas of particles is very diluted we can consider the two-particle structure as dominant and can approximate our system with a system of particles pairwise in contact interaction *in the background of the remaining a (N-2) -particle system*.

We have described contact interaction by Bethe-Peierls boundary conditions but we have not discussed the choice of the matrices $A_{i,j}$ for specific problems.

If the two-particle structure is dominant the matrix boundary matrix $A(y)$ is a function of the position of another particle.

In this case one can choose $A_{1,2} \equiv a(y)$ to be the probability that one of the other particles be in contact i.e $a(y) = c|\phi(y)|^2$.

Notice that we are in the regular case only if this term is sufficiently small.

As parameter of proportionality we can take the scattering length λ which does not depend on ϵ.

If the gas of particles is sufficiently dilute we can neglect the presence of the other particles; their only role is to permit the existence of the contact interaction of two particles.

The *linear* equation for each of the two particles in therefore

$$i\frac{\partial}{\partial t}\psi(x,t) = H_F\psi(x,t) - \lambda K(x,t)\psi(x,t) \qquad K(x,t) = |\phi(x,t)|^2 \tag{15}$$

where $\phi(x,t)$ is the wave function of the other particle.

If the function $|\phi(x,t)|$ is sufficiently regular the system is described at each time by a (time-dependent) self-adjoin operator.

If the particles in contact are identical this leads to consider the equation

$$i\frac{\partial}{\partial t}\psi(x,t) = (H_F\psi)(x,t) - \lambda|\psi(x,t)|^2\psi(x,t) \tag{16}$$

The solutions to this equation may therefore be regarded as describing, in the contact interaction limit, the solutions of a *linear equation for a pair of identical particles* in a bath of particles with which the two particles do non interact, under the constraint that the boundary condition for each of the two particles is given by the probability that the other be there.

For each of the two particles the function that describes the boundary condition (the effect of the other particle) is then precisely $|\psi(x,t)|^2$.

Recall that uniqueness of the self-adjoint extension holds only if the "charge" $|\psi(x)|^2$ is locally not too large. Otherwise one has infinitely many extensions.

This is reflected in the fact that the non liner equation has not a unique solution.

For larger densities the presence of other particles is felt and one has no longer a two-body problem.

We have seen that if one keeps the approximation of contact interactions it is sufficient to consider four particle subsystems.

The resonance lowers the energy of the system.

Therefore the two-body resonance enhances the binding energy between resonant pairs and reduces somehow the binding between non resonant pairs.

This may lead to a transition from a BEC (Bose Einstein Condensate) configuration (in which the binding within non resonating pairs is dominant) to a BCS (Bardeen, Cooper, Schrieffer) configuration in which binding of resonating pairs is increased.

This can be achieved by *a modulation of the (Feshbach) resonance*.

Added remarks

"Point interaction" and "contact interaction" are two different entities.

A relation between the two can be indicated as follows.

Since in point interactions a main role is taken by zero energy resonances , and since zero energy resonances play a role in a long time scale, it is convenient to consider large masses.

If one moreover wants to consider the case in which the positions of some of the particles can be regarded as fixed, one is led to consider these particles as even more massive.

Consider a $N+1$ particle systems in which particle A_0 has mass $\frac{1}{\lambda}$ and the remaining N particles $A_1, ..A_N$ have mass $\frac{1}{\lambda^2}$.

The particles interact trough two body potentials $V^{\epsilon}_{i,j}(|x_i - x_j|), i \neq j = 0 \dots N$ which scale as $V^{\epsilon}(|x|) = \frac{1}{\epsilon^3} V(\frac{|x|}{\epsilon})$

In the limit $\epsilon \to 0$ the particles interact by contact interactions.

For λ small the masses are very large and the limit $\epsilon \to 0$ has a unique solution.

Now let $\lambda = K\epsilon$ (K may be very large) .

Scale time by setting $\tau = \epsilon t$ so that on the τ time scale the displacement of particle A in unit time is of order one and for ϵ very small the particles B_i can be considered as located at fixed points $y_1 \cdots y_N$.

In this approximation and on the new time scale the Schrödinger equation for the particle under consideration reads

$$i\frac{\partial}{\partial t}\phi = (-K\Delta + \sum_i \frac{1}{\epsilon^2} V(\frac{x - y_i}{\epsilon^3}))\phi \tag{17}$$

In the limit $\epsilon \to 0$ this is the Schödinger equation for a particle of mass K subject to point interaction with N centers fixed at the point $y_1, \dots y_N$.

In the original time scale it is the Schrödinger equation for a particle of large mass interacting with N particles of much larger mass through a potential of very short range i.e. a contact interaction.

Therefore in this case point interaction can be regarded as an approximation to the asymptotic (in time) dynamics of a particle of large mass interacting with N particles of much larger mass through a potential of very short range.

Notice that after a time of order $\frac{1}{\epsilon}$ the motion is almost free (ballistic) if the two-body potentials have no zero energy resonance and is described by a point interactions if there is a zero energy resonance.

It follows that the $L^p \to L^p$ mapping properties of the Wave operators are different in the two cases [D] (and the difference is the same as in the case of regular potentials).

References

[A] S.Albeverio J. Operator Theory 6 (1981)

[A, K] S.Albeverio, P. Kurasov Lett. in Math. Physics 41 (1997) 79-92

[B] M.Birman Math. Sbornik. N.S. 38 (1956) 431-480

[B,F] Berezin-Faddaev Sov. Math. Doklady 2 (1961) 372-375

[B-P] Bethe-Peierls Proc of he Royal Academy of Sciences series A 148b (1935) 146-156

[Ba,P] B.Bazak, D.Petrov Phys. Rev Lett. 119 (2017) 083002-083006

[B,T] G.Basti, A.Teta arXiv 1601.08129 (2016)

[Br] A.Braides Weak convergence for beginners Oxford Univ. Press 2002

[C,M,P] Y. Castin, C.Mora, L.Pricoupenko PRL 105 (2016) 2232011-4

[C] M.Correggi et al Rev.Math.Phys. 24 (2012) 1250017-32

[C2] M.Correggi et al. Mathematical Physics, Analysis, Geometry 18(2016) 1–36

[D] G.F.Dell'Antonio et al accepted for publication in Annales Henry Poincar

[D,R] S.Derezinky, S.Richard Arkiv 1604 03340

[Da] G.S.Danilov Sov. Phys. JETP 13 (1961) 648-660)

[E] V.Efimov Nucler Physics A 210 (1971) 157-186

[F,T] D.Finco, A.Teta Rep. Math. Phys. 69 (2012) 131-159

[F] K. Friederichs K. Math. Annalen 109 (1934) 465-487

[K] M.G.Krein Math. Sbornik N.S. 20 (1962)1947 431-495

[Ko] S.Koshmanenko Acad. Sci. USSR Doklady 1988

[lY] A. le Yaouanc, L. Oliver, J-C Raynar J. of Mathematical Physics 38 (1997) 3998-4012

[M,P] A.Michelangeli, P.Pfeiffer J. PhysicsA 49 (2016) 105301-105331

[M1] R.Minlos Moscow Math. Journal 1 (2011) 111-127

[M2] R.Minlos Moscow Mat. Journal 14 (2014) 617-637

[M,F] R-Minlos, L.Faddeev Sov. Phys. Doklady 6 (1972) 1072-1074

[P] A.Posilicano Journ. Functional analysis 183 (2011) 109-147

[Pa] B.S.Pavlov Math. Sbornik 64 (1989) No 1

[Pe] D.S. Petrov Physcal Rev Letters 93 (2004) 143201- 143204

[S,T-M] K Ter-Martirosian , G Skorniakov Sov. Phys. JETP 4 (1956) 648-661

[vN] J von Neumann Math. Ann. 102(1930) 49-131

[W,C] F. Werner Y. Casten Phys. Rev. A 74 (2006) 053604